A general proof for the mean value theorem

3.37 $$\frac{dV_{ave}}{dR} = \underbrace{\frac{1}{4\pi R^2} \oint_{\substack{\text{sphere of}\\ \text{radius } R}} \nabla V \cdot d\vec{a}}_{\text{Right hand side (RHS)}} \longrightarrow \underline{\text{Prove}}$$

$$\text{RHS} = \frac{1}{4\pi R^2} \oint \nabla V \cdot d\vec{a}$$

$$\nabla V = \frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial T}{\partial \theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial T}{\partial \varphi}\hat{\varphi}$$

On the surface of the sphere $d\vec{a} = R^2 \sin\theta \, d\theta \, d\varphi \, \hat{r}$

$$\Rightarrow \nabla V \cdot d\vec{a} = \left(\frac{\partial V}{\partial r}\right)_{r=R} R^2 \sin\theta \, d\theta \, d\varphi$$

$$\Rightarrow \frac{1}{4\pi R^2} \oint \nabla V \cdot d\vec{a} = \frac{1}{4\pi} \iint \left(\frac{\partial V}{\partial r}\right)_{r=R} \sin\theta \, d\theta \, d\varphi$$

$$= \frac{\partial}{\partial r}\left[\frac{1}{4\pi} \iint V(R,\theta,\varphi) \sin\theta \, d\theta \, d\varphi\right]$$

$$= \frac{\partial}{\partial r} V_{ave}(R)\Big|_{r=R} = \frac{dV_{ave}}{dR}$$

<u>Also:</u> $\cancel{\oint_S (\nabla \cdot V) \cdot d\vec{a} = \oint_S}$ $\quad \oint_S \nabla V \cdot d\vec{a} = \int_{\mathcal{V}} \nabla \cdot (\nabla V) \, d\tau = \int_{\mathcal{V}} \nabla^2 V \, d\tau$

If V obey laplace equation $\Rightarrow \nabla^2 V = 0$

$$\Rightarrow \oint_S (\nabla V) \cdot d\vec{a} = 0 = \frac{dV_{ave}}{dR}$$

$\Rightarrow V_{ave}$ is independant of R.

$\Rightarrow V_{ave}(R) = \lim_{R \to 0} V_{ave}(R) = V_{ave}(0)$ = value V at the center of the sphere.