Stoke's law - A corollary

$$\int(\nabla'\times\vec{v})\cdot d\vec{a}' = \oint \vec{v}\cdot d\vec{l}' \rightarrow (1)$$

let $\vec{v} = \vec{c}\,T$ where $\vec{c}$ is a constant vector and T is a scalar (but not a constant). You can do this because $\vec{v}$ is <u>any</u> vector and $\vec{v} = \vec{c}\,T$ is a vector.

$$\nabla'\times\vec{v} = \nabla'\times(\vec{c}\,T) = T(\nabla'\times\vec{c}) - \vec{c}\times\nabla' T \rightarrow (2)$$

$\nabla'\times\vec{c} = 0$ because $\vec{c}$ is a constant vector. $\rightarrow (3)$

From (1), (2), (3)

$$\Rightarrow \int \nabla'\times(\vec{c}\,T)\cdot d\vec{a}' = -\int(\vec{c}\times\nabla' T)\cdot d\vec{a}' = \oint \vec{c}\cdot T\, d\vec{l}' \rightarrow (4)$$

[Recall scalar triple product: $\vec{A}\cdot(\vec{B}\times\vec{c}) = (\vec{A}\times\vec{B})\cdot\vec{c}$]

$$(4) \Rightarrow -\int(\vec{c}\times\nabla' T)\cdot d\vec{a}' = -\int\vec{c}\cdot(\nabla' T\times d\vec{a}') = \oint \vec{c}\cdot T\, d\vec{l}' \rightarrow (5)$$

$\vec{c}$ is a constant vector and can be taken out of the integral.

$$(5) \Rightarrow -\vec{c}\cdot\int \nabla' T\times d\vec{a}' = \vec{c}\cdot\oint T\, d\vec{l}'$$

$$\Rightarrow \quad -\int \nabla' T \times d\vec{a}' = \oint T \, d\vec{l}' \quad \rightarrow \quad (6)$$

let $T = \hat{r} \cdot \vec{r}'$ (6a) where $\hat{r}$ is a unit vector and a constant vector because it does not depend on the primed coordinate system i.e. (x', y', z') but:

$\hat{r} = \hat{r}_{x'} \hat{x}' + \hat{r}_{y'} \hat{y}' + \hat{r}_{z'} \hat{z}'$ (components in $\hat{x}', \hat{y}', \hat{z}'$ directions)

$$\Rightarrow \nabla' T = \nabla'(\hat{r} \cdot \vec{r}') = \hat{r} \times (\nabla' \times \vec{r}') + (\hat{r} \cdot \nabla')\vec{r}' \rightarrow (7)$$

$\nabla' \times \vec{r}'$ is zero ($\vec{r}'$ is the position vector).

$$(7) \Rightarrow \nabla' T = (\hat{r} \cdot \nabla')\vec{r}'$$

$$= \left(\hat{r}_{x'} \frac{\partial}{\partial x'} + \hat{r}_{y'} \frac{\partial}{\partial y'} + \hat{r}_{z'} \frac{\partial}{\partial z'}\right)(x'\hat{x}' + y'\hat{y}' + z'\hat{z}')$$

where we have used $\rightarrow$ $\hat{r} = \hat{r}_{x'} \hat{x}' + \hat{r}_{y'} \hat{y}' + \hat{r}_{z'} \hat{z}'$

$$\Rightarrow \nabla' T = \hat{r}_{x'} \hat{x}' + \hat{r}_{y'} \hat{y}' + \hat{r}_{z'} \hat{z}' = \hat{r} \rightarrow (8) \quad !!$$

$$* \; (6), (6a) \text{ and } (8) \Rightarrow -\int \hat{r} \times d\vec{a}' = \oint (\hat{r} \cdot \vec{r}') \, d\vec{l}' \rightarrow (9)$$

since $\hat{r}$ does not depend on the primed coordinates it can be taken out of surface integral

$$\Rightarrow \boxed{-\hat{r} \times \int d\vec{a}' = \oint (\hat{r} \cdot \vec{r}') \, d\vec{l}'}$$

(cannot take $\hat{r}$ out of the line integral because $(\hat{r} \cdot \vec{r}')$ operation has to be done first).