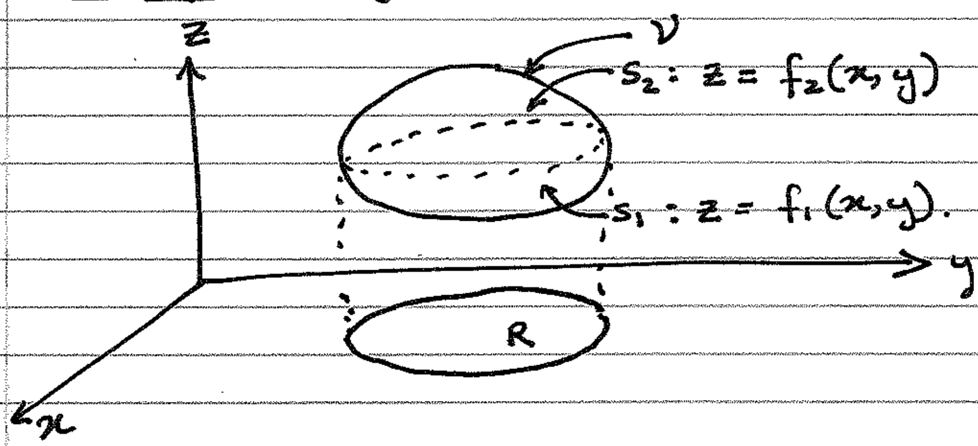
Proof of divergence theorem



The volume $\mathcal{V}$ is enclosed by the surface S.

The dotted line divides the volume $\mathcal{V}$ into two parts and S also into two parts S_1 & S_2 such that S_1 & S_2 ~~are single~~ can be expressed as single valued functions $S_1 : z = f_1(x,y)$ & $S_2 : z = f_2(x,y)$

Also: R is the projection of S on the x,y plane

let $\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z}$

$$\Rightarrow \nabla \cdot \vec{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

this is still a triple integral

To calculate $\iiint_{\mathcal{V}} (\nabla \cdot \vec{A})\, d\tau = \int_{\mathcal{V}} \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) d\tau$

first calculate: $\int_{\mathcal{V}} \frac{\partial A_z}{\partial z}\, d\tau = \int_{\mathcal{V}} \frac{\partial A_z}{\partial z}\, dx\, dy\, dz$

$$= \iint_R \left[\int_{f_1(x,y)}^{f_2(x,y)} \frac{\partial A_z}{\partial z}\, dz \right] dy\, dx.$$

Why does this work?

Consider an integral over a sphere of radius R_0

$$\int_v d\tau = \iint_{R_0} \left[\int_{f_1}^{f_2} dz \right] dy\, dx.$$

$z = f_1(x,y) = +\sqrt{R_0^2 - x^2 - y^2}$

& $z = f_2(x,y) = -\sqrt{R_0^2 - x^2 - y^2}$

$$= \iint_{R_0} \left(2\sqrt{R_0^2 - x^2 - y^2} \right) dy\, dx$$

in the circular projection of the sphere on the xy-plane (R_0) : $x^2 + y^2 = R_0^2$

Switching to polar coordinates for the circular projection

$$= 2 \iint \sqrt{R_0^2 - s^2}\; s\, ds\, d\varphi \qquad \begin{pmatrix} s^2 = x^2 + y^2 \\ da = s\, ds\, d\varphi \end{pmatrix}$$

$$= 2 \int_0^{2\pi} d\varphi \int_0^{R_0} \sqrt{R_0^2 - s^2}\; s\, ds \quad = \quad 4\pi \int_0^{R_0} \sqrt{R_0^2 - s^2}\; s\, ds$$

let $R_0^2 - s^2 = u \Rightarrow -2s\, ds = du \Rightarrow s\, ds = -\frac{du}{2}$

$\Rightarrow$ the integral $= -\dfrac{4\pi}{2}\displaystyle\int_{R_0^2}^{0}\sqrt{u}\,du = -2\pi \left.\dfrac{u^{3/2}}{3/2}\right|_{R_0^2}^{0}$

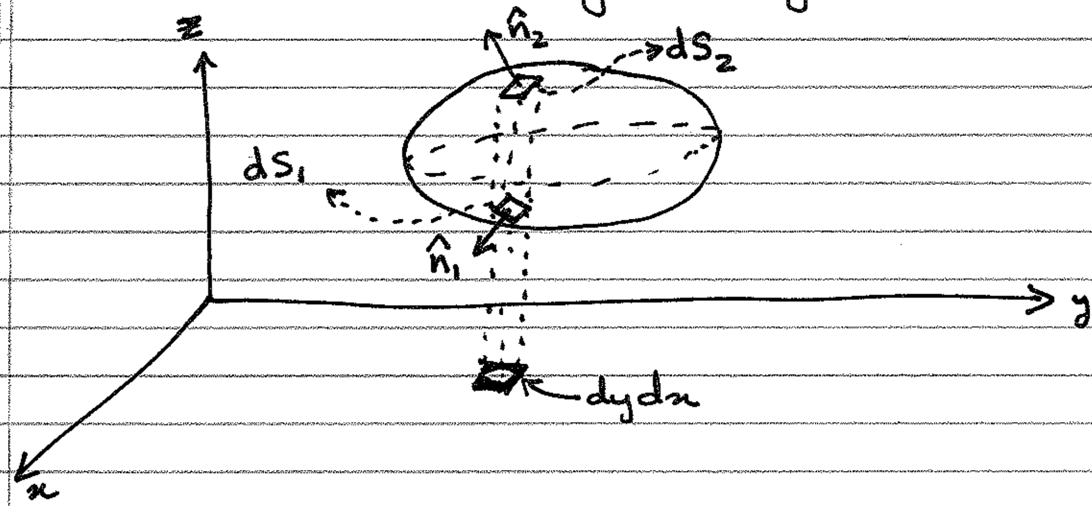
$= -\dfrac{4\pi}{3}\left[0-\left(R_0^2\right)^{3/2}\right] = \dfrac{4}{3}\pi R_0^3 \rightarrow$ volume of sphere !!

Going back to the original integral ...

$$= \iint_R \left[\int_{f_1}^{f_2} \frac{\partial A_z}{\partial z}\,dz\right] dy\,dx = \iint_R \left(A_z(x,y,z)\Big|_{f_1}^{f_2}\right) dy\,dx$$

$$= \iint_R \left[A_z(x,y,f_2) - A_z(x,y,f_1)\right] dy\,dx$$

Now back to the original figure ...



dydx is the projection of dS_2 on the xy-plane

$\Rightarrow$ $dydx = dS_2\,\hat{z}\cdot\hat{n}_2$ and $dydx = -dS_1\,\hat{z}\cdot\hat{n}_1$

For dS_1 the negative sign shows that $\hat{n}_1$ has a negative z-component.

$$\Rightarrow \iint_R \left[A_z(x, y, f_2) - A_z(x, y, f_1) \right] dy\,dx$$

$$= \iint_{S_2} A_z(x, y, z)\, \hat{z} \cdot \hat{n}_2\, ds_2 + \iint_{S_1} A_z(x, y, z)\, \hat{z} \cdot \hat{n}_1\, ds_1$$

$$= \iint_S A_z\, \hat{z} \cdot \hat{n}\, ds \quad = \iint_S A_z \hat{z} \cdot d\vec{S}$$

~~By~~ $$\Rightarrow \iiint_V \frac{\partial A_z}{\partial z}\, d\tau = \oiint_S A_z\, \hat{z} \cdot d\vec{S} \longrightarrow (1)$$

By projecting S on the yz & zx planes we can also obtain:

$$\iiint_V \frac{\partial A_x}{\partial x}\, d\tau = \oiint_S A_x\, \hat{x} \cdot d\vec{S} \longrightarrow (2)$$

and $$\iiint_V \frac{\partial A_y}{\partial y}\, d\tau = \oiint_S A_y\, \hat{y} \cdot d\vec{S} \longrightarrow (3)$$ respectively

Adding (1), (2), & (3):

$$\iiint_V \left(\frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z} \right) d\tau = \oiint_S \left(A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \right) \cdot d\vec{S}$$

$$\Rightarrow \iiint_V (\nabla \cdot \vec{A})\, d\tau = \oiint_S \vec{A} \cdot d\vec{S}$$