

$$\nabla^2 V = 0$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 V}{\partial \varphi^2} = 0$$

Azimuthal symmetry

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial V}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial V}{\partial \theta} \right) = 0$$

$$V(r, \theta) = R(r)\Theta(\theta)$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial R\Theta}{\partial r} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial R\Theta}{\partial \theta} \right) = 0$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \Theta \frac{dR}{dr} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( R \sin \theta \frac{d\Theta}{d\theta} \right) = 0$$

$$\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \Theta \frac{dR}{dr} \right) + \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} \left( R \sin \theta \frac{d\Theta}{d\theta} \right) = 0$$

$$\frac{\Theta}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{R}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = 0$$

Divide by  $R\Theta$

$$\frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{r^2 \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = 0$$

$$\frac{1}{Rr^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{r^2 \Theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = 0$$

$$\frac{1}{Rr^2} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{\Theta \sin^2 \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = 0$$

$f(R)$                        $g(\Theta)$

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = l(l+1)$$

$$\frac{1}{\Theta \sin^2 \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\Theta}{d\theta} \right) = -l(l+1)$$

# Laplace's equation in spherical coordinates

## The Radial part

$$\frac{1}{R} \frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) = l(l+1)$$

$$\Rightarrow r^2 \frac{d^2 R}{dr^2} + 2r \frac{dR}{dr} - l(l+1)R = 0$$

This diff. eq. has the property of increasing powers of  $r$  to compensate for higher derivatives of  $R$ . Such equations are called Euler-Cauchy equations and they have solutions

of the form  $R(r) = r^m$

$$\Rightarrow r^2 m(m-1) r^{m-2} + 2r m r^{m-1} - l(l+1) r^m = 0$$

$$\Rightarrow [m(m-1) + 2m - l(l+1)] r^m = 0$$

$$\Rightarrow m^2 - m + 2m - l(l+1) = 0$$

$$\Rightarrow m^2 + m - l(l+1) = 0, \quad m = \frac{-1 \pm \sqrt{1 + 4l(l+1)}}{2}$$
$$= \frac{-1 \pm \sqrt{4l^2 + 4l + 1}}{2} = \frac{-1 \pm (2l+1)}{2}$$

$$\Rightarrow m = l, -(l+1)$$

$$\Rightarrow R(r) = r^l \text{ or } r^{-(l+1)} \Rightarrow R(r) = A r^l + \frac{B}{r^{l+1}}$$

## The angular part

$$\frac{1}{\theta \sin \theta} \frac{d}{d\theta} \left( \sin \theta \frac{d\theta}{d\theta} \right) = -l(l+1)$$

$$\begin{aligned} \text{let } \cos \theta = x &\Rightarrow \frac{dx}{d\theta} = -\sin \theta = -\sqrt{1-x^2} \quad \text{and} \quad \frac{d}{d\theta} = \frac{d}{dx} \frac{dx}{d\theta} \\ &\Rightarrow \sin \theta = \sqrt{1-x^2} \end{aligned}$$

$$\Rightarrow \frac{1}{\theta \sqrt{1-x^2}} \frac{d}{dx} \left( \sqrt{1-x^2} \frac{d\theta}{dx} \frac{dx}{d\theta} \right) \frac{dx}{d\theta} = -l(l+1)$$

$$\Rightarrow \frac{-\sqrt{1-x^2}}{\theta \sqrt{1-x^2}} \frac{d}{dx} \left[ -(1-x^2) \frac{d\theta}{dx} \right] = -l(l+1)$$

$$\Rightarrow \frac{d}{dx} \left[ (1-x^2) \frac{d\theta}{dx} \right] = -l(l+1) \theta$$

$$\Rightarrow -2x \frac{d\theta}{dx} + (1-x^2) \frac{d^2\theta}{dx^2} = -l(l+1) \theta$$

$$\Rightarrow \boxed{(1-x^2) \frac{d^2\theta}{dx^2} - 2x \frac{d\theta}{dx} + l(l+1)\theta = 0} \quad \text{Legendre's equation} \rightarrow \textcircled{1}$$

$$\text{let } \theta = \sum_{m=0}^{\infty} a_m x^m \rightarrow \textcircled{2}$$

$$\Rightarrow (1-x^2) \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - 2x \sum_{m=1}^{\infty} m a_m x^{m-1} + l(l+1) \sum_{m=0}^{\infty} a_m x^m = 0$$

$$\Rightarrow \sum_{m=2}^{\infty} m(m-1) a_m x^{m-2} - \sum_{m=2}^{\infty} m(m-1) a_m x^m - 2 \sum_{m=1}^{\infty} m a_m x^m + l(l+1) \sum_{m=0}^{\infty} a_m x^m = 0$$

Now we need to collect terms with the same power of  $x$ :

$$x^0: l(l+1)a_0 + 2a_2 \rightarrow \textcircled{3}$$

$$x^1: l(l+1)a_1 - 2a_1 + 6a_3 \rightarrow \textcircled{4}$$

for  $m \geq 2$ :

$$x^m: l(l+1)a_m - 2ma_m - m(m-1)a_m + (m+2)(m+1)a_{m+2}$$

$$= \underbrace{[l(l+1) - m - m^2]}_{=} a_m + (m+2)(m+1)a_{m+2}$$

$$= l^2 + l - m - m^2 = l^2 - m^2 + l - m$$

$$= (l-m)(l+m) + (l-m)$$

$$= (l-m)(l+m+1)$$

$$\rightarrow = (l-m)(l+m+1)a_m + (m+2)(m+1)a_{m+2} \rightarrow \textcircled{5}$$

If  $\textcircled{2}$  is to be a solution of diff. eq.  $\textcircled{1}$  for all  $x$

then all the coefficients of  $x^m$  ( $m=0 \dots \infty$ ) ~~have~~ in equations  $\textcircled{3}$ ,  $\textcircled{4}$ , &  $\textcircled{5}$  have to individually go to zero.

$$\Rightarrow l(l+1)a_0 + 2a_2 = 0$$

$$\Rightarrow a_2 = -\frac{l(l+1)}{2}a_0$$

$$\Rightarrow l(l+1)a_1 - 2a_1 + 6a_3 = 0$$

$$\Rightarrow a_3 = -\frac{(l^2 + l - 2)a_1}{6} = -\frac{(l+2)(l-1)a_1}{6}$$

and:

$$\Rightarrow (l-m)(l+m+1)a_m + (m+2)(m+1)a_{m+2} = 0$$

$$\Rightarrow \boxed{a_{m+2} = -\frac{(l-m)(l+m+1)a_m}{(m+2)(m+1)}} \rightarrow \text{recursion formula.} \rightarrow \textcircled{6}$$

This formula gives all  $a_m$  except  $a_0$  &  $a_1$ , which are left as arbitrary constants.

If  $l$  is even i.e.  $0, 2, 4, \dots$

then above a certain value of  $m$ , all  $a_m$  go to zero for even  $m$ .

e.g. if  $l=2$ ,  $a_0 \neq 0$  but  $a_2 \neq 0$ , but  $a_4 = 0$  because of the  $l-m$  term in equation  $\textcircled{6}$ .  
Hence,  $a_6, a_8, \dots$  are all zero.

let  $a_0 = 1$  (arbitrary constant and 1 is easy to handle)

Then for  $l=0$ , the solution is just  $\Theta = \sum_{m=0,2,\dots}^{\infty} a_m x^m$

$$\Rightarrow \Theta = 1$$

because  $a_2 = 0$  (using equation (6)) and so are

$a_4, a_6, \dots$

We are ignoring the odd coefficients because they add up to unphysical solutions.

For  $l=2$ ,  $a_0 = 1$ ,  $a_2 = -3a_0 = -3$ , &  $a_4, a_6, \dots = 0$

$$\Rightarrow \Theta = a_0 x^0 + a_2 x^2 = 1 - 3x^2$$

This solution is usually written as:  $\frac{1}{2}(3x^2 - 1)$

to account for normalization.

Similarly for odd  $l$ , let  $a_1 = 1$

For  $l=1$ ,  $a_3, a_5, \dots = 0$

$$\Rightarrow \Theta = a_1 x^1 = x$$

for  $l=3$ ,  $a_1 = 1$ ,  $a_3 = -\frac{5}{3}a_1 = -\frac{5}{3}$  &  $a_5, a_7, \dots = 0$

$$\Rightarrow \Theta = a_1 x^1 + a_3 x^3 = x - \frac{5}{3}x^3$$

This solution is written as:  $\frac{1}{2}(5x^3 - 3x)$  for normalization

Collecting all the circled polynomials:

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

⋮

These are called Legendre polynomials ~~and are~~

$P_l(x)$  and these are solutions to the diff. eq.

$$(1-x^2) \frac{d^2\theta}{dx^2} - 2x \frac{d\theta}{dx} + l(l+1)\theta = 0$$

⇒ for the original equation:

$$\frac{1}{\theta \sin\theta} \frac{d}{d\theta} \left( \sin\theta \frac{d\theta}{d\theta} \right) = -l(l+1)$$

The solutions are  $P_l(\cos\theta)$ .

We ignore non-positive, non-integer  $l$  and the infinite series solution due to either the given boundary conditions or because they provide unphysical solutions which blow up.