$\mathcal{L}$1

- Variational problem [6.2] find a solution $y(x)$ for which the integral S is stationary

$$S = \int_{x_1}^{x_2} f(y(x), y'(x), x)\, dx$$

$y(x_1) = y_1 \quad y(x_2) = y_2$

- The Euler-Lagrange Equation (ELE) [6.2]

$$\frac{\partial f}{\partial y} - \frac{d}{dt}\frac{\partial f}{\partial y'} = 0$$

differential equation for a stationary path $y(x)$

- Lagrangian $\mathcal{L}$ [Ch 7] for a mechanical system $\mathcal{L} = T - U$

T = kinetic energy U = potential energy

- Hamilton principle [Ch 7]

action $S = \int_{t_1}^{t_2} \mathcal{L}\, dt$ is stationary ~~when~~ when integral S is taken along the actual path of the particle.

- the ELE defines the equation of motion (Newton's Laws)

Example: one particle in cartesian coordinates

- generalized coordinates, degrees of freedom
- ignorable or cyclic coordinates
  - conservation Laws
    * generalized force: $F_i = \frac{\partial \mathcal{L}}{\partial q_i}$
    * generalized momentum: $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$

(ELE) $F_i = \frac{d}{dt} p_i$

- Symmetry → conservation law

(not any symmetry results in conservation law, though)

L 1

- Hamiltonian: $H = \sum_i p_i \dot{q}_i - L$

when L does not depend explicitely on t
H is conserved: $\frac{dH}{dt} = 0$

- For natural generalized coordinates $H = T + U$
(that $r_\alpha(q_1, \dots q_n)$ is time independent)
r_α – cartesian coordinates α - particle index
q_i – generalized coordinates. i - coordinate index

$$\dot{r}_\alpha = \sum_i \frac{\partial r_\alpha}{\partial q_i} \dot{q}_i \qquad \dot{r}_\alpha^2 = \sum_{ij} \underbrace{\frac{\partial r_\alpha}{\partial q_i} \frac{\partial r_\alpha}{\partial q_j}}_{A_{ij}(q)} \dot{q}_i \dot{q}_j$$

$$T = \frac{1}{2} \sum_\alpha m_\alpha \dot{r}_\alpha^2 = \frac{1}{2} \sum_{\alpha ij} m_\alpha A_{\alpha ij} \dot{q}_i \dot{q}_j = \frac{1}{2} \sum A_{ij} \dot{q}_i \dot{q}_j$$

$$p_i = \frac{\partial L}{\partial \dot{q}_i} = \frac{\partial T}{\partial \dot{q}_i} = \sum_j A_{ij} \dot{q}_j \, ;$$

$$\sum p_i \dot{q}_i = \sum_i \sum_j A_{ij} \dot{q}_i \dot{q}_j = 2T$$

$\Rightarrow$ $H = \sum p_i \dot{q}_i - L = T + U$

- Hamilton Equations. (just for 1 DOF) [4|3]

* $L = T - U = \frac{1}{2} A(q) \dot{q}^2 - U(q)$
* $p = \frac{\partial L}{\partial \dot{q}} = A(q) \dot{q} \xrightarrow[\text{for } \dot{q}]{\text{Solve}} \dot{q} = \frac{1}{A(q)} p = \dot{q}(p, q)$
* $H(p,q) = p \dot{q}(p,q) - L(q, \dot{q}(p,q))$
* $\frac{\partial H}{\partial q} = p \frac{\partial \dot{q}}{\partial q} - \frac{\partial L}{\partial q} - \frac{\partial L}{\partial \dot{q}} \frac{\partial \dot{q}}{\partial q} = -\frac{\partial L}{\partial q} = -\dot{p}$
* $\frac{\partial H}{\partial p} = \left(\dot{q} + p \frac{\partial \dot{q}}{\partial p}\right) - \underbrace{\frac{\partial L}{\partial \dot{q}}}_{p} \frac{\partial \dot{q}}{\partial p} = \dot{q}$

- Harmonic oscillator $H = \frac{m \dot{x}^2}{2} + \frac{1}{2} k x^2$
$p = m \dot{x} \qquad q = x \qquad H = p^2/2m + \frac{1}{2} k q^2$

$$\frac{\partial H}{\partial q} = k q = -\dot{p} \qquad \frac{\partial H}{\partial p} = \frac{p}{m} = \dot{q} \rightarrow \ddot{q} + \frac{k}{m} q = 0$$