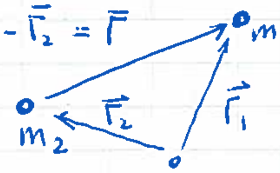
## L2 Two-body central-force problem

$\vec r_1 - \vec r_2 = \vec r$




$$|F| = \frac{G m_1 m_2}{|r|^2} = \frac{K}{r^2}$$

$$U = -\frac{G m_1 m_2}{|\vec r|} = -\frac{K}{r}$$

same potential for gravity & EM with different K: $K_G$ $K_{EM}$

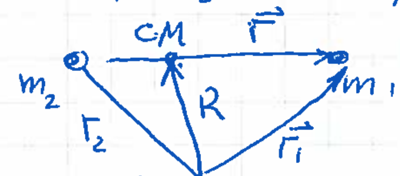
$$\mathcal{L} = \frac{1}{2} m_1 \dot{\vec r}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec r}_2^{\,2} - U(r)$$

- Center of mass: $\vec R = \dfrac{\vec r_1 m_1 + \vec r_2 m_2}{m_1 + m_2} = \dfrac{m_1 \vec r_1 + m_2 \vec r_2}{M}$

$M = m_1 + m_2$ – total mass

$P = M \dot R$ – total momentum

$P = \text{const.}$




$$\vec r_1 = \vec R + \frac{m_2}{M} \vec r \qquad \vec r_2 = \vec R - \frac{m_1}{M} \vec r;$$

$$T = \frac{1}{2} m_1 \dot{\vec r}_1^{\,2} + \frac{1}{2} m_2 \dot{\vec r}_2^{\,2} = \frac{1}{2} m_1 \left(\dot{\vec R} + \frac{m_2}{M} \dot{\vec r}\right)^2 + \frac{1}{2} m_2 \left(\dot{\vec R} - \frac{m_1}{M} \dot{\vec r}\right)^2$$

$$= \frac{1}{2} M \dot{\vec R} + \frac{1}{2} \frac{m_1 m_2}{M} \dot{\vec r}^{\,2} = \frac{1}{2} M \dot{\vec R}^2 + \frac{1}{2} \mu \dot{\vec r}^{\,2}$$

$$\mu = \frac{m_1 m_2}{m_1 + m_2} \quad \text{– reduced mass}$$

$$\mu_{m_1 \gg m_2} \approx m_1 \qquad \boxed{T = \frac{1}{2} M \dot{\vec R}^2 + \frac{1}{2} \mu \dot{\vec r}^{\,2}}$$

$$\mathcal{L} = \frac{1}{2} M \dot{\vec R}^2 + \frac{1}{2} \mu \dot{\vec r}^{\,2} - U(r)$$

- The equations of motion

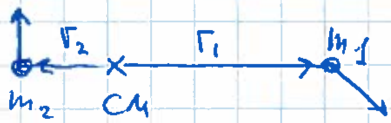
* $\dfrac{\partial \mathcal{L}}{\partial R} = 0 = \dfrac{d}{dt} \dfrac{\partial \mathcal{L}}{\partial \dot R} = M \ddot R = \dfrac{d}{dt}(M \dot R) \to M \dot R = \text{const.}$

  conservation of total $\vec P$

* $\dfrac{\partial \mathcal{L}}{\partial r} = -\vec\nabla U(r) = \mu \ddot{\vec r}$ → motion of particle $\mu$ in potential $U(r)$

L2

- CM reference frame $\rightarrow \dot{R} = 0$




$$\mathcal{L} = \frac{1}{2}\mu \dot{\vec{r}}^2 - U(r)$$

case $m_2 \gg m_1$ $\rightarrow$ CM is at particle 2

- Conservation of angular momentum

$\vec{L} = \vec{r}_1 \times \vec{p}_1 + \vec{r}_2 \times \vec{p}_2$ – angular momentum

in CM frame: $\vec{r}_1 = \frac{m_2}{\mu}\vec{r}$ $\quad \vec{r}_2 = \frac{m_1}{\mu}\vec{r}$

$$\vec{L} = \frac{m_1 m_2}{M}\left( m_2 (\vec{r} \times \dot{\vec{r}}) + m_1 \vec{r} \times \dot{\vec{r}} \right) = \vec{r} \times \mu \dot{\vec{r}}$$

$\vec{L}$ is the angular momentum of a "single" particle $\mu$ at distanse $\vec{r}$ and velocity $\dot{\vec{r}}$

$$\frac{d\vec{L}}{dt} = \frac{d}{dt}(\vec{r} \times \mu \dot{\vec{r}}) = \underbrace{\dot{\vec{r}} \times \mu \dot{\vec{r}}}_{=0} + \underbrace{\vec{r} \times \mu \ddot{\vec{r}}}_{\vec{r} \times \vec{F} = 0}$$

$\vec{L}$ is conserved.

- the equations of motion.

use polar coordinates: $r_x = r\cos\varphi$, $r_y = r\sin\varphi$

$$\mathcal{L} = \frac{1}{2}\mu \dot{\vec{r}}^2 - U(r) = \frac{1}{2}\mu(\dot{r}_x^2 + \dot{r}_y^2) - U(r)$$

$$= \frac{1}{2}\mu(\dot{r}^2 + r^2\dot{\varphi}^2) - U(r)$$

* $\frac{\partial \mathcal{L}}{\partial \varphi} = 0 \rightarrow \frac{\partial \mathcal{L}}{\partial \dot{\varphi}} = \mu r^2 \dot{\varphi} = |L| = \text{const} = L_z$

* radial equation: $\frac{\partial \mathcal{L}}{\partial r} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{r}}$

$$\mu r \dot{\varphi}^2 - \frac{dU}{dr} = \mu \ddot{r} \qquad \dot{\varphi} = \frac{L_z}{\mu r^2}$$