L4 Orbits

- radial equation: $\mu\ddot{r} = -\frac{dU}{dr} + \mu r\dot{\varphi}$; using $\dot{\varphi} = \frac{\ell}{\mu r^2}$

$$\mu\ddot{r} = F_G(r) + \ell^2/\mu r^3 \rightarrow r(t)$$

How to find $r(\varphi)$ — the orbit; $r = \frac{1}{u}$

u is a new variable $= 1/r$ to express

$$\frac{d}{dt} = \frac{d\varphi}{dt}\frac{d}{d\varphi} = \frac{\ell}{\mu r^2}\frac{d}{d\varphi} = \frac{\ell u^2}{\mu}\frac{d}{d\varphi}$$

$$\dot{r} = \frac{\ell u^2}{\mu}\frac{d}{d\varphi}\left(\frac{1}{u}\right) = -\frac{\ell}{\mu}\frac{du}{d\varphi}; \quad \frac{d}{d\varphi}\left(\frac{1}{u}\right) = -\frac{1}{u^2}\frac{du}{d\varphi}$$

$$\ddot{r} = \frac{d}{dt}(\dot{r}) = \frac{\ell u^2}{\mu}\frac{d}{d\varphi}\left(-\frac{\ell}{\mu}\frac{du}{d\varphi}\right) = -\frac{\ell^2 u^2}{\mu^2}\frac{d^2u}{d\varphi^2}$$

- radial equatio (u)

$$-\frac{\ell^2 u^2}{\mu^2}\mu\frac{d^2u}{d\varphi^2} = F_G(r) + \frac{\ell^2 u^3}{\mu} \rightarrow u''(\varphi) = -u(\varphi) - \frac{\mu F(u)}{\ell^2 u^2(\varphi)}$$

for a given force $F_G(u)$ this diff. eq. gives $u(\varphi)$

- $F(u) = 0 \qquad u''(\varphi) = -u(\varphi) \qquad u(\varphi) = A\cos(\varphi - \delta)$

or using $A = \frac{1}{r_0} \rightarrow r(\varphi) = \frac{r_0}{\cos(\varphi-\delta)}$

straight line in polar coordinates.

- Kepler Orbits: $F(u) = -ku^2 = -\frac{k}{r^2}$

$$u''(\varphi) = -u(\varphi) + k\mu/\ell^2 = -w(\varphi) = w''(\varphi)$$

$$w(\varphi) = A\cos(\varphi-\delta) = u(\varphi) - k\mu/\ell^2$$

$$u(\varphi) = A\cos(\varphi-\delta) + k\mu/\ell^2 = \frac{k\mu}{\ell^2}(1+\varepsilon\cos(\varphi-\delta))$$

the orbit: $r(\varphi) = \dfrac{\ell^2/k\mu}{1+\varepsilon\cos(\varphi-\delta)} = \dfrac{c}{1+\varepsilon\cos(\varphi-\delta)}$

L4 Bounded orbits ($\varepsilon < 1$)

$r_{min} = \dfrac{c}{1-\varepsilon}$ $\qquad r_{max} = \dfrac{c}{1+\varepsilon}$

perihelion $\varphi = 0$ $\qquad$ aphelion $\varphi = \pi$

$r(2\varphi) = r(0)$ – closed orbit.

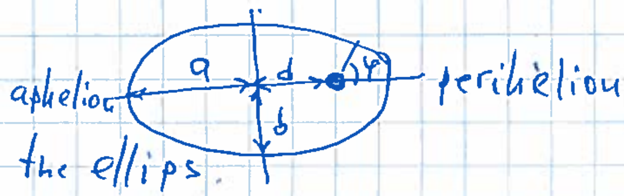
- in cartesian coordinates $r-(x,y)$

$x = r\cos\varphi \qquad y = r\sin\varphi$

$$r(\varphi) = \frac{c}{1+\varepsilon\cos\varphi} \rightarrow \left(\frac{x+d}{a}\right)^2 + \left(\frac{y}{b}\right)^2 = 1 \text{ (ellips)}$$

$$a = \frac{c}{1-\varepsilon^2} \qquad b = \frac{c}{\sqrt{1-\varepsilon^2}} \qquad d = a\varepsilon$$

$$\frac{b}{a} = \sqrt{1-\varepsilon^2}$$



ε – eccentricity of the ellips.

- Kepler's third Law: $dA = \frac{1}{2}|\vec{r} \times \vec{v}\,dt] \Rightarrow$

$$\frac{dA}{dt} = \frac{1}{2m}|\vec{r}\times\vec{p}] = \frac{1}{2m}\ell \rightarrow \frac{1}{2\mu}\ell \text{ for 2 body problem.}$$

Area of the ellips $A = \pi a b$; $\tau = \dfrac{A}{dA/dt} = \dfrac{2\pi a b \mu}{\ell}$

$$\tau^2 = 4\pi^2 \frac{a^3 c\mu^2}{\ell^2} = 4\pi^2 \frac{a^3\mu^2}{\ell^2}\cdot\frac{\ell^2}{k\mu} = 4\pi^2\frac{a^3\mu}{k};$$

$$k = G m_1 m_2 = G\mu M \approx G\mu M_s$$

$$\tau^2 = \frac{4\pi^2}{GM_s}a^3 \text{ – the third Kepler's law.}$$

- $E(\varepsilon)$ – energy $E = U_{eff}(r_{min}) = -\dfrac{k}{r_{min}} + \dfrac{\ell^2}{2\mu r_{min}^2}$

$$r_{min} = \frac{c}{1+\varepsilon} \rightarrow E = \frac{k^2\mu}{2\ell^2}(\varepsilon^2 - 1)$$

$\varepsilon > 1 \rightarrow E > 0$ – unbounded; $\varepsilon < 1 \rightarrow E < 0$ – bounded.