L10 Rotation of Rigid Bodies.

A rigid body is a collection of (N) particles which shape does not change.

3N generalized coordinates → 6 coordinates.

center of Mass: 3   orientation 3

- Center of mass

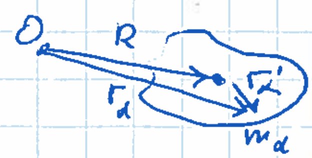
$$\vec R = \frac{1}{M}\sum_\alpha m_\alpha r_\alpha \;;\; M = \sum_\alpha m_\alpha \implies$$

$$\vec R = \frac{1}{M}\int_V \vec r\, dm \;;\quad M = \int_V dm \qquad \int_V \text{- integral over volume.}$$

momentum: $\vec P = \sum_\alpha m_\alpha \dot{\vec r}_\alpha = M\dot{\vec R}$; $\vec F_{ext} = M\ddot{\vec R}$

- Agular momentum.

L_O - angular momentum relative to origin O



$\vec r_\alpha = \vec R + \vec r_\alpha'$

$$\vec \ell_\alpha = \vec r_\alpha \times \vec p_\alpha = \vec r_\alpha \times m\dot{\vec r}_\alpha \qquad L_O = \sum \ell_\alpha = \sum_\alpha \vec r_\alpha \times m\dot{\vec r}_\alpha$$

$$L_O(R, r') = \sum_\alpha R\times m_\alpha \dot R + \sum_\alpha R\times m_\alpha \dot{\vec r}_\alpha' + \sum_\alpha \vec r_\alpha' \times m_\alpha \dot{\vec R} + \sum_\alpha \vec r_\alpha' \times m_\alpha \dot{\vec r}_\alpha'$$

$$= R\times M\dot R + R\times \underbrace{\sum m_\alpha \dot{\vec r}_\alpha'}_{0} + \underbrace{\left(\sum m_\alpha \vec r_\alpha'\right)}_{0}\times R + \sum \vec r_\alpha' \times m_\alpha \dot{\vec r}_\alpha'$$

$0 = \frac{1}{M}\sum m_\alpha \vec r_\alpha'$ is a position of CM in CM frame

$$L_O = L_{orb} + L_{spin} = \vec R \times M\dot{\vec R} + \sum_\alpha \vec r_\alpha' \times m_\alpha \dot{\vec r}_\alpha'$$

- torque: $\dot L_{orb} = \underbrace{\dot{\vec R}\times m\dot{\vec R}}_{0} + \vec R\times M\ddot{\vec R} = \vec R\times \vec F_{ext} \to 0$ for central Force

$$\dot L = \vec\Gamma_{ext} = \sum r_\alpha \times F_\alpha^{ext} = \underbrace{\sum \vec r_\alpha' \times \vec F_{ext}}_{L_{spin}} + \underbrace{R\times \vec F_{ext}}_{L_{orb}}$$

$$\dot L_{spin} = \sum \vec r_\alpha' \times \vec F_{ext} = \vec\Gamma_{ext} \text{ (about CM)}.$$

Earth – Sun: $\dot L_{spin} \sim \varepsilon(t) \sim 0$ because of the shape of the Earth & orbit.

p – e orbita moment & spin.

210 Kinetic Energy

$$T = \sum_\alpha m_\alpha \frac{1}{2}\dot{\vec r}_\alpha^{\,2} = \frac{1}{2}\sum_\alpha m_\alpha \underset{\uparrow \atop \text{cm}}{\dot{\vec R}^2} + \frac{1}{2}\sum \underset{\uparrow \atop \text{relative to cm (rotation)}}{m_\alpha \dot{\vec r}'^{\,2}_\alpha} + \dot{\vec R}\,\overset{0}{\overset{\|}{\sum m_\alpha \dot{\vec r}'_\alpha}}$$

$\vec R$ can be for any point fixed to the body.
if we select a point which is at rest $\dot{\vec R} = 0$

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{\vec r}'^{\,2}_\alpha$$

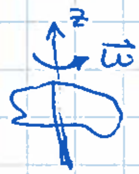

rolling wheel

$\dot{\vec R} = 0$

Potential energy

$$U = U_{ext} + U_{int} \qquad U_{int} = \sum_{\alpha<\beta} U_{\alpha\beta}(r_{\alpha\beta}) = \text{const.} = 0$$ sele

– fixed axis



$\hat\omega = \hat z$

$$\vec L = \sum \vec r_\alpha \times m \vec v_\alpha$$

$$\vec v_\alpha = \vec\omega \times \vec r_\alpha \qquad \vec\omega = (0,0,\omega) \qquad \vec r_\alpha = (x_\alpha, y_\alpha, z_\alpha)$$

$$\vec\omega \times \vec r_\alpha = \begin{vmatrix} \hat x & \hat y & \hat z \\ 0 & 0 & \omega \\ x_\alpha & y_\alpha & z_\alpha \end{vmatrix} = (-\omega y_\alpha, \omega x_\alpha, 0)$$

$$m\, \vec r_\alpha \times \vec\omega \times \vec r_\alpha = \begin{vmatrix} \hat x & \hat y & \hat z \\ x_\alpha & y_\alpha & z_\alpha \\ -\omega y_\alpha & \omega x_\alpha & 0 \end{vmatrix} = m_\alpha \omega \left(-z_\alpha x_\alpha, -z_\alpha y_\alpha, x_\alpha^2 + y_\alpha^2\right)$$

$$L_z = \sum m_\alpha (x_\alpha^2 + y_\alpha^2)\,\omega = \sum m_\alpha \rho_\alpha^2 \omega = I_z \omega$$

I_z: moment of inertia about z axis

$$T = \frac{1}{2}\sum_\alpha m_\alpha v_\alpha^2 = \frac{1}{2}\sum_\alpha m_\alpha \rho_\alpha^2 \omega^2 = \frac{1}{2} I_z \omega^2$$

$$L_x = -\sum m_\alpha x_\alpha z_\alpha \omega \qquad L_y = -\sum m_\alpha y_\alpha z_\alpha \omega$$

$L_x \neq 0$ $L_y \neq 0$ $\rightarrow$ $\vec L$ can be pointing in different direction than $\vec\omega$