L11 Products of Inertia.

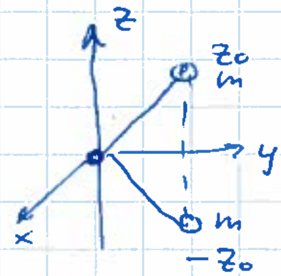
$$L_x = -\sum m_\alpha x_\alpha z_\alpha \omega_z = I_{xz}\,\omega_z$$ – x component of L when ω is in z direct.

$$L_y = -\sum m_\alpha y_\alpha z_\alpha \omega_z = I_{yz}\,\omega_z$$

$$L_z = \sum m_\alpha (x_\alpha^2 + y_\alpha^2)\,\omega_z = I_{zz}\,\omega_z$$

I_{xz}, I_{yz}, I_{zz} – products of inertia.

Example I



$x_\alpha = 0 \rightarrow I_{xz} = 0$

$I_{yz} = y_0 z_0 - y_0 z_0 = 0$

$I_{zz} = 2m y_0^2$

Rotation around arbitrary axis

$\vec{\omega} = (0, 0, \omega_z) \rightarrow (\omega_x, \omega_y, \omega_z)$

from superposition of angular momentum.

$$L_x = I_{xz}\,\omega_z + I_{xy}\,\omega_y + I_{xx}\,\omega_x$$

$I_{xz} = -\sum m_\alpha x_\alpha z_\alpha \quad I_{xy} = -\sum m_\alpha x_\alpha y_\alpha \quad I_{xx} = \sum m_\alpha (y_\alpha^2 + z_\alpha^2)$

$$L_y = I_{yx}\,\omega_x + I_{yy}\,\omega_y + I_{yz}\,\omega_z$$

$I_{yx} = -\sum m_\alpha y_\alpha x_\alpha \;;\; I_{yy} = \sum m_\alpha (x_\alpha^2 + z_\alpha^2) \;;\; I_{yz} = -\sum m_\alpha y_\alpha z_\alpha$

$$L_z = I_{zx}\,\omega_x + I_{zy}\,\omega_\alpha + I_{zz}\,\omega_z$$

$I_{zx} = -\sum m_\alpha z_\alpha x_\alpha \;;\; I_{zy} = -\sum m_\alpha z_\alpha y_\alpha \quad I_{zz} = \sum m_\alpha (x_\alpha^2 + y_\alpha^2)$

$$I = \begin{vmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{vmatrix} = \begin{vmatrix} \sum m_\alpha\, y_\alpha^2 + z_\alpha^2 & -\sum m_\alpha x_\alpha y_\alpha & -\sum m_\alpha x_\alpha z_\alpha \\ -\sum m_\alpha y_\alpha x_\alpha & \sum m_\alpha (x_\alpha^2 + z_\alpha^2) & -\sum m_\alpha y_\alpha z_\alpha \\ -\sum m_\alpha z_\alpha x_\alpha & -\sum m_\alpha z_\alpha y_y & \sum m_\alpha (x_\alpha^2 + y_\alpha^2) \end{vmatrix}$$

$\vec{a} \times (\vec{b} \times \vec{c}) = \vec{b}(\vec{a}\cdot\vec{c}) - \vec{c}(\vec{a}\cdot\vec{b})$ abc rule

L 11

$$L = I\,\omega \rightarrow \begin{vmatrix} L_x \\ L_y \\ L_z \end{vmatrix} = I \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

I is symmetric matrix : $I_{ij} = I_{ji}$
or $I = \tilde{I}$ – transpose matrix.

* Inertia tensor of a solid cube

– rotation around it's corner : $I_{xx} = I_{yy} = I_{zz}$

$$I_{xx} = \int_0^a dx \int_0^a dy \int_0^a dz\, \rho(x^2+y^2) \leftarrow \sum_\alpha m_\alpha (x_\alpha^2 + y_\alpha^2)$$

$$m_\alpha \rightarrow \rho\, dx\, dy\, dz \qquad \rho = \frac{M}{a^3}$$

$$I_{xx} = I_{xx}^x + I_{xx}^y \qquad I_{xx}^y = I_{xx}^x$$

$$I_{xx}^x = \frac{M}{a^3}\int_0^a x^2 dx \int_0^a dy \int_0^a dz = \frac{M}{a^3}\cdot\frac{1}{3}a^3\cdot a\cdot a = \frac{1}{3}Ma^2$$

$$I_{xx} = \frac{2}{3}Ma^2$$

$$I_{xy} = I_{xz} = I_{yz} \qquad I_{xy} = -\int_0^a dx \int_0^a dy \int_0^a dz \frac{M}{a^3}\, xy$$

$$I_{xy} = -\int_0^a x\,dx \int_0^a y\,dy \int_0^a dz \frac{M}{a^3} = -\frac{a^2}{2}\cdot\frac{a^2}{2}\cdot a\cdot\frac{M}{a^3} = -\frac{1}{4}Ma^2$$

$$L = \begin{vmatrix} L_x \\ L_y \\ L_z \end{vmatrix} = I = \frac{Ma^2}{12}\begin{vmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{vmatrix} \times \omega = \begin{vmatrix} \omega_x \\ \omega_y \\ \omega_z \end{vmatrix}$$

rotation around x axis $\quad \omega = (\omega, 0, 0)$

$$L = I\begin{pmatrix} \omega \\ 0 \\ 0 \end{pmatrix} = \frac{Ma^2}{12}\omega\,(8, -3, -3) \qquad L \nparallel \omega$$

* rotation around it's main diagonal : $\vec{a} = \alpha(1,1,1)$

$$\hat{a} = \frac{1}{\sqrt{3}}(1,1,1) \qquad \vec{\omega} = |\omega|\,\hat{u}$$

$$L = I\omega = \frac{Ma^2}{12}\frac{\omega}{\sqrt{3}}\begin{vmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{vmatrix}\begin{vmatrix} 1 \\ 1 \\ 1 \end{vmatrix} = \frac{Ma^2}{12}\frac{\omega}{\sqrt{3}}\begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \frac{Ma^2}{6}\vec{\omega}$$

$L \parallel$

Cube rotating around its center.

$$I^{x}_{xx} = \frac{M}{a^3} \int_{-a/2}^{a/2} x^2 dx \int_{-a/2}^{a/2} dy \int_{-a/2}^{a/2} dz = \frac{M}{a^3} \frac{1}{3} 2 \cdot \frac{a^3}{8} \cdot 2\frac{a}{2} \cdot 2 a/2 = \frac{M}{12} a^2$$

$$I_{xx} = \frac{M}{6} a^2$$

$$I_{xy} = -\underbrace{\int_{-a/2}^{a/2} x\, dx}_{=0} \underbrace{\int_{-a/2}^{a/2} y\, dy}_{=0} \int dz \frac{M}{a^3} = 0$$

$$I = \frac{Ma^2}{6} \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = \frac{Ma^2}{6} 1$$ — often use "I"

— Principal Axes

when $\vec{L} \sim \vec{\omega}$ we call rotation axis principal

Therefore there are 3 principal axes:

$$I = \begin{vmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{vmatrix} \quad \text{so} \quad L = I\omega \quad L \parallel \omega$$

$\lambda_1\ \lambda_2\ \lambda_3$ — principal moments

Kinetic Energy $T = \frac{1}{2} \omega L = \frac{1}{2}\left[\lambda_1 \omega_1^2 + \lambda_2 \omega_2^2 + \lambda_3 \omega_3^2\right]$