L12

Principal axes/moments

- principal axis: $\vec{L} = \lambda \vec{\omega}$ $\quad \vec{L} \parallel \vec{\omega}$
- principal moment: λ - moment of Inertia about the axis defined by ω

- Example 10.2 for rotation axis $\frac{1}{\sqrt{3}}(1,1,1)$ about the corner of the cube $\lambda = \frac{1}{6} M a^2$
  Are there other principal axes?
- Example 10.2 - rotation about the center
  I is diagonal $\rightarrow$ there are 3 principal axes with the same $\lambda_1 = \lambda_2 = \lambda_3 = \frac{1}{6} M a^2$

- in general there could be 3 principal axes with $\lambda_1 \neq \lambda_2 \neq \lambda_3$

- Important: The existance of PA does not assume a particular symmetry of a rigid body
  For any rigid dody & any rotation point O there are 3 orthogonal principal axes.
  If $\vec{\omega}$ is pointing along any of these axes, $\vec{L} \parallel \vec{\omega}$
- This result is general: for any symmetric matrix I (real) there is a set of orthogonal axes (with the same origin) for which I' is diagonal. - this is a theorem.

- how to find principal axes/moments.
  Given I (3x3 matrix) we need to find.
  vectors $\vec{\omega}_1, \vec{\omega}_2, \vec{\omega}_3$ such that
  $\{ I\vec{\omega}_1 = \lambda_1 \vec{\omega}_1 \quad I\vec{\omega}_2 = \lambda_2 \vec{\omega}_2 \quad I\vec{\omega}_3 = \lambda_3 \vec{\omega}_3 \} \Rightarrow I\vec{\omega} = \lambda \vec{\omega}$
  $\vec{\omega}_i$ - is eigenvector | the procedure can be
  λ_i - is eigenvalue | generalized for any symmetric matrix I (not only 3x3)
- how to find $\lambda_i, \vec{\omega}_i$? firs find λ_i
- characteristic equation
  $(I - \lambda u)\omega = 0 \rightarrow \det(I - \lambda u) = 0$ $\quad u$ - unit matrix.

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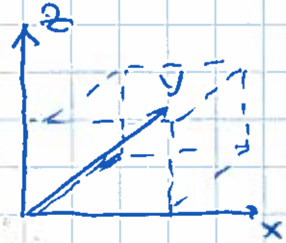
$$\det(I - \lambda u) = 0$$

For 3X3 Inertia matrix (often called tensor) $\det(I - \lambda u) = 0$ is a cubic equation with solutions $\lambda_1, \lambda_2, \lambda_3$

Example 10.4 Principal axes of a cube around the corner.



for selected axes: x, y, z,

$$I = \underbrace{\frac{Ma^2}{2}}_{\mu} \begin{vmatrix} 8 & -3 & -3 \\ -3 & 8 & -3 \\ -3 & -3 & 8 \end{vmatrix}$$

by rotating $xyz \to x', y', z'$ we want to find anothe set of orthogohal axes x', y', z' where I is diagonal

$$\det(I - \lambda u) = \det \begin{vmatrix} 8\mu - \lambda & -3\mu & -3\mu \\ -3\mu & 8\mu - \lambda & -3\mu \\ -3\mu & -3\mu & 8\mu - \lambda \end{vmatrix} = 0 =$$

$$= (2\mu - \lambda)(11\mu - \lambda)^2 ; \quad \lambda_1 = 2\mu ; \ \lambda_2 = \lambda_3 = 11\mu ;$$

— Now find eigenvectors

$$(I - \lambda_1 u)\omega = 0 \to \mu \begin{vmatrix} 6 & -3 & -3 \\ -3 & 6 & -3 \\ -3 & -3 & 6 \end{vmatrix} \begin{vmatrix} \omega_x \\ \omega_y \\ \omega_z \end{vmatrix} = 0$$

$\omega_x = \omega_y = \omega_z$

$\omega_1 = \frac{1}{\sqrt{3}}(1, 1, 1)$

$$(I - \lambda_2 u)\omega = 0 \to 3\mu \begin{pmatrix} -1 & -1 & -1 \\ -1 & -1 & -1 \\ -1 & -1 & -1 \end{pmatrix} \begin{vmatrix} \omega_x \\ \omega_y \\ \omega_z \end{vmatrix} = 0$$

$\omega_x + \omega_y + \omega_z = 0$

$\vec{\omega} \cdot \vec{e}_1 = 0$

$\vec{\omega}_2 \perp \vec{\omega}_3$ and $\perp \vec{\omega}_1$ if we selech coordinat frame $\vec{e}_1, \vec{e}_2, \vec{e}_3$ than

$$I = \begin{vmatrix} \lambda_1 & 0 \\ 0 & \ddots \end{vmatrix}$$