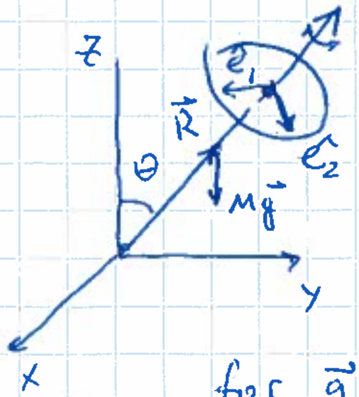
L13 Precession of rotating top.



$\vec{\omega} = \omega \hat{e}_3$

$$\bar{I} = \begin{vmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \lambda_3 \end{vmatrix}$$

$\lambda_1 = \lambda_2$ because of axial symmetry in $\hat{e}_1, \hat{e}_2, \hat{e}_3$ frame.

Consider $\vec{g} = 0 \rightarrow \vec{L} = \lambda_3 \omega \hat{e}_3$

for $\vec{g} \neq 0$ there is torque $\vec{\Gamma} = \vec{R} \times M\vec{g}$

$\vec{\Gamma} = \mathring{\vec{L}}$

$|\Gamma| = RMg \sin\theta$

if $\vec{\Gamma}$ is small $\vec{L}$ slowly changes

assuming $\vec{\Gamma} \perp \hat{z}$ $\vec{\Gamma} \perp \hat{e}_3 \rightarrow$ only the direction of L changes.

$$\lambda_3 \omega \mathring{\hat{e}}_3 = \vec{R} \times M\vec{g} = -M\vec{g} \times \vec{R}$$

$$\mathring{\hat{e}}_3 = \frac{MgR}{\lambda_3 \omega} \hat{z} \times \hat{e}_3 = \vec{\Omega} \times \vec{e}_3 \qquad \vec{\Omega} = \frac{MgR}{\lambda_3 \omega} \hat{z}$$

the axis of the top rotates about $\hat{z}$ axis with the angular velocity Ω

- Euler's equation(s)

"space frame"

$x, y, z,$ - inertial frame

"body frame"

$e_1 e_2 e_3$ - non-inertial frame fixed to the body.

in the body frame $\vec{L} = \{\lambda_1 \omega_1, \lambda_2 \omega_2, \lambda_3 \omega_3\}$

$$\left(\frac{d\vec{L}}{dt}\right)_{space} = \vec{\Gamma} = \left(\frac{dL}{dt}\right)_{frame} + \vec{\omega} \times \vec{L} = \dot{\vec{L}} + \vec{\omega} \times \vec{L}$$

$$\left.\begin{aligned} \Gamma_1 &= \lambda_1 \dot{\omega}_1 - (\lambda_2 - \lambda_3)\omega_2\omega_3 \\ \Gamma_2 &= \lambda_2 \dot{\omega}_2 - (\lambda_3 - \lambda_1)\omega_3\omega_1 \\ \Gamma_3 &= \lambda_3 \dot{\omega}_3 - (\lambda_1 - \lambda_2)\omega_1\omega_2 \end{aligned}\right\} \text{Euler's equation(s)}$$

- free precession ($\vec{\Gamma} = 0$) & $\lambda_1 = \lambda_2 = \lambda$

$$\Omega_b = \frac{\lambda - \lambda_3}{\lambda} \omega_3$$

$$\lambda \mathring{\omega}_1 = (\lambda - \lambda_3)\omega_2\omega_3 \Rightarrow \mathring{\omega}_1 = \Omega_b \omega_2$$

$$\lambda \mathring{\omega}_2 = (\lambda_3 - \lambda)\omega_1\omega_3 \qquad \mathring{\omega}_2 = -\Omega_b \omega_1$$

$$\lambda_3 \mathring{\omega}_3 = 0 \qquad \omega_3 = \text{const.}$$

L13 Free precession with $\lambda_1 = \lambda_2 = \lambda$.

$$\omega_3 = \text{const}$$
$$\dot\omega_1 = \Omega_b\,\omega_2 \qquad \Omega_b = \left(1 - \frac{\lambda_3}{\lambda}\right)\omega_3 \qquad \eta = \omega_1 + i\omega_2$$
$$\dot\omega_2 = -\Omega_b\,\omega_1$$

$$\dot\eta = -i\Omega_b\eta \to \eta = \eta_0 e^{-i\Omega_b t}$$

$$|\eta| = \eta_0 = \underset{\text{const.}}{\underset{\shortparallel}{\omega_0}} \qquad \vec\omega = (\omega_0\cos\Omega_b t, -\omega_0\sin\Omega_b t, \omega_3)$$

$(\vec\omega \cdot \hat e_3) = |\omega|\cos\alpha = \sqrt{\omega_0^2 + \omega_3^2}\cos\alpha$ $\quad\alpha$ - const.

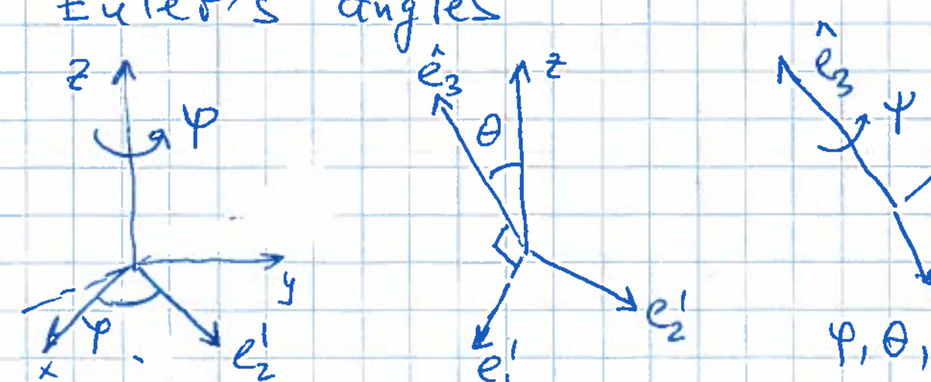
in the body frame $(\hat e_1, \hat e_2, \hat e_3)$ $\vec\omega$ moves around the cone  $\leftarrow$ the body cone

$$\vec L = (\lambda_1\omega_0\cos\Omega_b t, -\lambda_1\omega_0\sin\Omega_b t, \lambda_3\omega_3)$$

$\vec L$ & $\vec\omega$ precess around $\hat e_3$ with the angular velocity $\Omega_b = \left(1 - \frac{\lambda_3}{\lambda}\right)\omega_3$

Top is precessing if even $\vec\Gamma = 0$

- Euler's angles



See Figure 10.10 in the textbook

φ, θ, ψ - Euler's angles

$\hat x\,\hat y\,\hat z \to \varphi, \theta, \psi \to \hat e_1, \hat e_2, \hat e_3$, $\qquad \vec\omega = \dot\varphi\hat z + \dot\theta\hat e_2' + \dot\psi\hat e_3$

$\lambda_1 = \lambda_2$, $\qquad \hat z = \hat e_3\cos\theta - \hat e_1'\sin\theta$

$$\vec\omega = -\dot\varphi\sin\theta\,\hat e_1' + \dot\theta\,\hat e_2' + (\dot\psi + \dot\varphi\cos\theta)\,\hat e_3$$
$$\vec L = -\lambda_1\dot\varphi\sin\theta\,\hat e_1' + \lambda_1\dot\theta\,\hat e_2' + \lambda_3(\dot\psi + \dot\varphi\cos\theta)\,\hat e_3$$

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$$\lambda_1 = \lambda_2 = \lambda$$

$$T = \frac{1}{2}\lambda\left(\dot\varphi^2 \sin^2\theta + \dot\theta^2\right) + \frac{1}{2}\lambda_3\left(\dot\psi + \dot\varphi\cos\theta\right)^2$$

$$\mathcal{L} = T - MgR\cos\theta$$

$$\theta: \quad \lambda\ddot\theta = \lambda\dot\varphi^2\cos\theta\sin\theta - \lambda_3(\dot\psi + \dot\varphi\cos\theta)\dot\varphi\sin\theta + MgR\sin\theta$$

$$\varphi: \quad \lambda\dot\varphi\sin^2\theta + \lambda_3(\dot\psi + \dot\varphi\cos\theta)\cos\theta = \text{const.} = P_\varphi$$

$$\psi: \quad \lambda_3(\dot\psi + \dot\varphi\cos\theta) = \text{const} = P_\psi$$

$$L_z = (\vec{L}\cdot\hat{z}) = (\vec{L},\ \cos\theta\,\hat{e}_3 - \sin\theta\,\hat{e}_1') =$$

$$\lambda\dot\varphi\sin^2\theta + \underbrace{\lambda_3(\dot\psi + \dot\varphi\cos\theta)}_{L_3}\cos\theta = L_z$$

$$\dot\varphi = \frac{L_z - L_3\cos\theta}{\lambda\sin^2\theta} \to \text{const for } \theta = \text{const.}$$