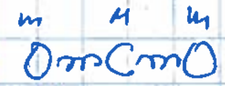
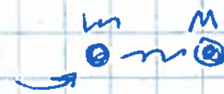
L15 Coupled oscillators

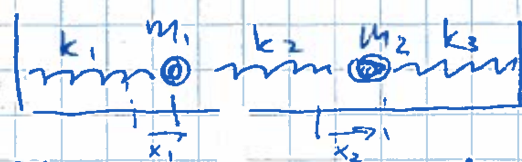
oscillations of several bodies (masses) interacting with each other.

For example : CO_2; $O = C = O$ ⇒ 

Simple oscillator → 

coupled oscillator.

2 masses & 3 springs.



a) $k_2 = 0$ (no spring) 2 independent oscillators.

$$F_1 = -k_1 x_1 + k_2(x_2 - x_1) = -(k_1+k_2)x_1 + k_2 x_2$$

$$F_2 = -k_3 x_2 + k_2(x_1 - x_2) = -(k_2+k_3)x_2 + k_2 x_1$$

$$X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad M = \begin{vmatrix} m_1 & 0 \\ 0 & m_2 \end{vmatrix} \qquad K = \begin{vmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2+k_3 \end{vmatrix}$$

$$z_1 = x_1(t) + i\,y_1(t) = \alpha_1\left(\cos(\omega t - \delta_1) + i\sin(\omega t - \delta_2)\right) = a_1 e^{i\omega t}$$

$$z_2 = x_2(t) + i\,y_2(t) = \alpha_2\left(\cos(\omega t - \delta_2) + i\sin(\omega t - \delta_2)\right) = a_2 e^{i\omega t}$$

$$\vec{z}(t) = \vec{a}\,e^{i\omega t} = \begin{vmatrix} a_1 \\ a_2 \end{vmatrix} e^{i\omega t} = \begin{vmatrix} \alpha_1 e^{-i\delta_1} \\ \alpha_2 e^{-i\delta_2} \end{vmatrix} e^{i\omega t}$$

complex.

$$\vec{X}(t) = \begin{vmatrix} x_1 \\ x_2 \end{vmatrix} = \mathrm{Re}\,\vec{z}(t)$$

$$M\ddot{X} = -KX \rightarrow -\omega^2 M a\, e^{i\omega t} = -K a\, e^{i\omega t}$$

$$(K - \omega^2 M)\,a = 0 \qquad \text{if } \det(K - \omega^2 M) \neq 0 \quad a = 0$$

$$\det(K - \omega^2 M) = 0$$ quadratic equations for ω → ω_1, ω_2

ω_1 ω_2 – normal frequencies

oscillation with ω_1, ω_2 – normal modes.

L15

$k_1 = k_2 = k_3 \qquad m_1 = m_2$

$$M = \begin{vmatrix} m & 0 \\ 0 & m \end{vmatrix} \qquad K = \begin{vmatrix} 2k & -k \\ -k & 2k \end{vmatrix} \qquad (K - \omega^2 M) = \begin{vmatrix} 2k - m\omega^2 & -k \\ -k & 2k - m\omega^2 \end{vmatrix}$$

$$\det(K - \omega^2 M) = (k - \omega^2 m)(3k - \omega^2 m)$$

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{\frac{3k}{m}}$$

- first normal mode: $\begin{vmatrix} k & -k \\ -k & k \end{vmatrix} \begin{vmatrix} a_1 \\ a_2 \end{vmatrix} = 0 \Rightarrow \begin{matrix} a_1 - a_2 = 0 \\ -a_1 + a_2 = 0 \end{matrix} \quad a_1 = a_2$

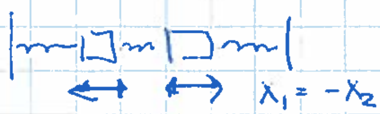
$z(t) = A \begin{vmatrix} 1 \\ 1 \end{vmatrix} e^{i(\omega_1 t - \delta_1)} \qquad x(t) = Re(z(t)) = A \begin{vmatrix} 1 \\ 1 \end{vmatrix} \cos(\omega_1 t - \delta_1)$

carts oscillate in phase and not interacting.

- second normal mode: $\begin{vmatrix} -k & -k \\ -k & -k \end{vmatrix} \begin{vmatrix} a_1 \\ a_2 \end{vmatrix} \Rightarrow a_1 = -a_2$

$z(t) = A \begin{vmatrix} 1 \\ -1 \end{vmatrix} e^{i(\omega_2 t - \delta)} \rightarrow x(t) = A \begin{vmatrix} 1 \\ -1 \end{vmatrix} \cos(\omega_2 t - \delta_2)$

carts oscillate exactly out of phase



- general motion is superposition of two solutions $\quad X(t) = A_1 \begin{vmatrix} 1 \\ 1 \end{vmatrix} \cos(\omega_1 t - \delta_1) + A_2 \begin{vmatrix} 1 \\ -1 \end{vmatrix} \cos(\omega_2 t - \delta_2)$

since $\omega_2 = \sqrt{3}\,\omega_1$ the motion of each cart never repeats itself — not quasiperiodic !

- normal coordinates: $x_1(t), x_2(t) \rightarrow \xi_1(t), \xi_2(t)$

$$\begin{matrix} m\ddot{x}_1 = -2k x_1 + k x_2 \\ m\ddot{x}_2 = k x_1 - 2k x_2 \end{matrix} \Rightarrow \begin{matrix} m\ddot{\xi}_1 = -k\xi_1 \\ m\ddot{\xi}_2 = -3k\xi_2 \end{matrix} \quad \text{normal coordinates}$$

$$\xi_1 = \frac{x_1 + x_2}{2} \qquad \xi_2 = \frac{x_1 - x_2}{2} \qquad K = \begin{pmatrix} k & 0 \\ 0 & 3k \end{pmatrix}$$

$$(K - \omega^2 M)\, a = 0 \quad \rightarrow a_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} A\cos(\omega_1 t - \delta) \quad a_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} A\cos(\omega_2 t - \delta$$

L15 Weakly Coupled oscillators.

$$K = \begin{vmatrix} k+k_2 & -k_2 \\ -k_2 & k+k_2 \end{vmatrix}$$

$$\omega_1 = \sqrt{\frac{k}{m}} \qquad \omega_2 = \sqrt{\frac{k+2k_2}{m}} \qquad \omega_1 \sim \omega_2 \quad k_2 \ll k.$$

$$\omega_1 = \omega_0 - \varepsilon \qquad \omega_2 = \omega_2 + \varepsilon \qquad \omega_0 = \frac{\omega_1+\omega_2}{2}$$

2 normal modes $\quad z_1 = A_1 \begin{pmatrix}1\\1\end{pmatrix} e^{i(\omega_0-\varepsilon)t} \quad z_2 = A_2 \begin{pmatrix}1\\-1\end{pmatrix} e^{i(\omega_0+\varepsilon)t}$

$$Z(t) = Z_1 + Z_2 = e^{i\omega_0 t}\left[A_1\begin{pmatrix}1\\1\end{pmatrix} e^{-i\varepsilon t} + A_2 \begin{pmatrix}1\\-1\end{pmatrix} e^{i\varepsilon t}\right]$$

Case: $A_1 = A_2 = A/2$

$$Z(t) = \frac{A}{2} e^{i\omega_0 t}\left[\begin{matrix} e^{-i\varepsilon t} + e^{+i\varepsilon t} \\ e^{-i\varepsilon t} - e^{+i\varepsilon t}\end{matrix}\right] = A e^{i\omega_0 t}\begin{pmatrix}\cos\varepsilon t \\ -i\sin\varepsilon t\end{pmatrix}$$

$$\mathrm{Re}(Z) = \begin{matrix} x_1 = A\cos\varepsilon t\,\cos(\omega_0 t) \\ x_2 \quad A\sin\varepsilon t\,\cos(\omega_0 t) \end{matrix}$$

- beats: $\quad t=0 \quad x_1 = A \quad x_2 = 0 \quad \dot{x}_1 = \dot{x}_2 = 0$

$$\left.\begin{matrix} t \approx 0 & x_1(t) \approx A\cos\omega_0 t \\ & x_2(t) \approx 0 \\ t = \pi/2\varepsilon & x_1(t) \approx 0 \\ & x_2(t) \approx A\sin\omega_0 t \end{matrix}\right\} \text{beats.}$$

$$\xi_1 = \frac{x_1+x_2}{2} = A\cos(\omega_0-\varepsilon)t = A\cos\omega_1 t$$

$$\xi_2 = \frac{x_1-x_2}{2} = A\cos(\omega_0+\varepsilon)t = A\cos\omega_2 t$$

normal coordinates oscillate with $\omega_1\ \omega_2$
their suprposition experience beats — constructive & distructive interference.