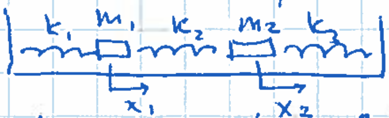
L16 Lagrangian approach to coupled oscillators

- 

$$\mathcal{L} = T - U$$

$$T = \frac{1}{2} m_1 \dot{x}_1^2 + \frac{1}{2} m_2 \dot{x}_2^2$$

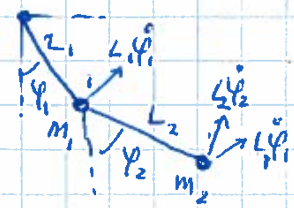
$$U = \frac{1}{2} k_1 x_1^2 + \frac{1}{2} k_2 (x_2 - x_1)^2 + \frac{1}{2} k_3 (-x_2)^2$$

$$\frac{\partial \mathcal{L}}{\partial x_1} = -\left(k_1 x_1 + k_2 (x_1 - x_2)\right) = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_1} = m \ddot{x}_1 = -(k_1 + k_2) x_1 + k_2 x_2$$

$$\frac{\partial \mathcal{L}}{\partial x_2} = -\left(k_3 x_2 + k_2 (x_2 - x_1)\right) = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_2} = m \ddot{x}_2 = -k_2 x_1 - (k_2 + k_3) x_2$$

$$M \ddot{\vec{x}} = -K \vec{x} \qquad x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

- Double Pendulum



$$V_1 = L_1 \dot{\varphi}_1 \qquad V_2 = L_2 \dot{\varphi}_2$$

$$U_1 = m_1 g L_1 (1 - \cos\varphi_1)$$

$$U_2 = m_2 g \left[L_2 (1 - \cos\varphi_2) + L_1 (1 - \cos\varphi_1) \right]$$

$$U = (m_1 + m_2) g L_1 (1 - \cos\varphi_1) + m_2 g L_2 (1 - \cos\varphi_2)$$

$$T_1 = \frac{1}{2} m_1 L_1^2 \dot{\varphi}_1^2 \qquad T_2 = \frac{1}{2} m_2 (\vec{V}_1 + \vec{V}_2)^2 = \frac{1}{2} m_2 \left[V_1^2 + 2 \vec{V}_1 \vec{V}_2 + V_2^2 \right]$$

$$|V_1|^2 = L_1^2 \dot{\varphi}_1^2 \qquad |V_2|^2 = L_2^2 \dot{\varphi}_2^2 \qquad \vec{V}_1 \cdot \vec{V}_2 = L_1 L_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)$$

small oscillations: $U(\varphi_1, \varphi_2) = \underbrace{U(0)}_{\text{const}} + \underbrace{\frac{\partial U}{\partial \varphi_1}\varphi_1 + \frac{\partial U}{\partial \varphi_2}\varphi_2}_{=0} + \underbrace{\sum \frac{\partial U^2}{\partial \varphi_i \partial \varphi_j} \varphi_i \varphi_j}_{\sim \varphi^2} + \ldots$

$1 - \cos\varphi \to \varphi^2/2$

$$T_{12} = m_2 L_1 L_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2) \approx m_2 L_1 L_2 \dot{\varphi}_1 \dot{\varphi}_2 \sim \dot{\varphi}^2$$

$$\mathcal{L} = \frac{1}{2}(m_1 + m_2) L_1^2 \dot{\varphi}_1^2 + \frac{1}{2} m_2 L_2^2 \dot{\varphi}_2^2 + m_2 L_1 L_2 \dot{\varphi}_1 \dot{\varphi}_2 - \frac{1}{2}(m_1 + m_2) g L_1 \varphi_1^2 - m_2 g L_2 \varphi_2^2$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_1} = \frac{\partial \mathcal{L}}{\partial \varphi_1} \to (m_1 + m_2) L_1^2 \ddot{\varphi}_1 + m_2 L_1 L_2 \ddot{\varphi}_2 = -(m_1 + m_2) g L_1 \varphi_1$$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\varphi}_2} = \frac{\partial \mathcal{L}}{\partial \varphi_2} \to m_2 L_1 L_2 \ddot{\varphi}_1 + m_2 L_2^2 \ddot{\varphi}_2 = -m_2 g L_2 \varphi_2$$

L16 $\qquad M\ddot{\varphi} = -k\varphi$

$$M = \begin{vmatrix} (m_1+m_2)L_1^2 & m_2 L_1 \\ m_2 L_1 L_2 & m_2 L_2^2 \end{vmatrix} \quad K = \begin{vmatrix} (m_1+m_2) g L_1 & 0 \\ 0 & m_2 g L_2 \end{vmatrix} \quad \varphi = \begin{vmatrix} \varphi_1 \\ \varphi_2 \end{vmatrix}$$

search for a solution $\varphi(t) = Re\, z(t) \quad \vec{z}(t) = \vec{a} e^{i\omega t}$

$$(Ka - \omega^2 M a) = (K - \omega^2 M) a = 0 \to \det (K - \omega^2 M) = 0$$

- $m_1 = m_2 = m \qquad L_1 = L_2 = L \qquad \omega_0^2 = g/L \quad g = L\omega_0^2$

$$M = mL^2 \begin{vmatrix} 2 & 1 \\ 1 & 1 \end{vmatrix} \quad K = mL^2 \begin{vmatrix} 2\omega_0^2 & 0 \\ 0 & \omega_0^2 \end{vmatrix} \to mL^2 \begin{vmatrix} 2(\omega_0^2 - \omega^2) & -\omega^2 \\ -\omega^2 & \omega_0^2 - \omega^2 \end{vmatrix}$$

$$\det (K - \omega M) = mL^2 \left[2(\omega_0^2 - \omega^2)^2 - \omega^4 \right] = \omega^4 - 4\omega^2\omega_0^2 + 2\omega_0^2 = 0$$

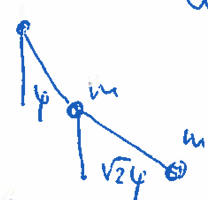
$$\omega^2 = (2 \pm \sqrt{2})\,\omega_0^2 \qquad \omega_1^2 = (2-\sqrt{2})\,\omega_0^2 \qquad \omega_2^2 = (2+\sqrt{2})\,\omega_0^2$$

- normal modes.

I) $(K - \omega_1^2 M)a = mL^2 \omega_0^2 (\sqrt{2}-1) \begin{vmatrix} 2 & -\sqrt{2} \\ -\sqrt{2} & 1 \end{vmatrix} \begin{vmatrix} a_1 \\ a_2 \end{vmatrix} = 0 \qquad a_2 = \sqrt{2}\, a_1$

$a_1 = A_1 e^{-i\delta_1}$

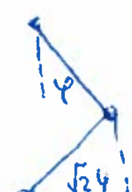
$$\varphi_a = \begin{vmatrix} \varphi_1 \\ \varphi_2 \end{vmatrix}_a = A \begin{vmatrix} 1 \\ \sqrt{2} \end{vmatrix} \cos(\omega_1 t - \delta_1)$$



II) $(K - \omega_2^2 M) b = mL^2 \omega_0^2 (\sqrt{2}+1) \begin{vmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 1 \end{vmatrix} \begin{vmatrix} b_1 \\ b_2 \end{vmatrix} = 0 \qquad b_2 = -\sqrt{2}\, b_1$

$b_1 = B e^{-i\delta_2}$

$$\varphi_b = \begin{vmatrix} \varphi_1 \\ \varphi_2 \end{vmatrix}_b = B \begin{vmatrix} 1 \\ -\sqrt{2} \end{vmatrix} \cos(\omega_2 t - \delta_2)$$



$$\varphi(t) = \varphi_a + \varphi_b = A \begin{vmatrix} 1 \\ \sqrt{2} \end{vmatrix} \cos(\omega_1 t - \delta_1) + B \begin{vmatrix} 1 \\ -\sqrt{2} \end{vmatrix} \cos(\omega_2 t - \delta_2)$$