L17 Coupled oscillator with many degrees of freedom.

generalized coordinates $\vec{q} = (q_1, q_2, \dots q_n)$

Conservative system: $U(q_1 \dots q_n) = U(\vec{q})$ - potential energy.

Kinetic energy: $T = \sum_\alpha \frac{1}{2} m_\alpha \dot{\vec{r}}_\alpha^2$ where

$\vec{r}_\alpha = r_\alpha(q_1 \dots q_n)$ — does not explicitely depends on time. $\rightarrow \dot{r}_\alpha^2 = \sum_i \frac{\partial r_\alpha}{\partial q_i}\dot{q}_i \cdot \sum_j \frac{\partial r_\alpha}{\partial q_j}\dot{q}_j =$

$$T = \frac{1}{2}\sum_\alpha m_\alpha \dot{r}_\alpha^2 = \frac{1}{2}\sum_{ij} A_{ij}(\vec{q})\,\dot{q}_i \dot{q}_j$$

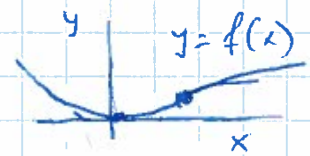
$M_{ij} = A_{ij}(0)$; $A_{ij}(q) = \sum_\alpha m_\alpha \frac{\partial r_\alpha}{\partial q_i}\frac{\partial r_\alpha}{\partial q_j}$ - only function of $\vec{q}$

$\vec{q} = 0$ - the equilibrium point (just select so)

$$U(q) = \underset{\substack{\parallel \\ \text{const}}}{U(0)} + \sum_j \underset{\substack{\parallel \\ 0}}{\frac{\partial U}{\partial q_j}} q_j + \underbrace{\frac{1}{2}\sum_{ij}\frac{\partial^2 U}{\partial q_i \partial q_j} q_i q_j}_{\sim q^2} + \dots$$

$$U(q) \approx \frac{1}{2}\sum_{ij}\frac{\partial^2 U}{\partial q_i \partial q_j} q_i q_j = \frac{1}{2}\sum K_{ij} q_i q_j$$

Example 11.1



$U = mg f(x)$

$$U = mg f(0) + mg \underset{\substack{\parallel \\ 0}}{f'(0)} + mg\frac{1}{2} f''(0) x^2$$

$\dot{y} = f'(x)\dot{x}$

$$T = \frac{1}{2} m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2} m \dot{x}^2 (1 + f'^2(x)) \approx \frac{1}{2} m \dot{x}^2$$

$$\mathcal{L} = \frac{1}{2} m \dot{x}^2 - mg\frac{1}{2} f''(0) x^2 \Rightarrow m\ddot{x} = -mg f''(0) x$$

$$\omega^2 = g f''(0)$$

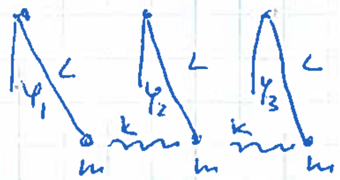
- General equation of motion: $\mathcal{L} = \frac{1}{2}\sum_{ij} M_{ij}\dot{q}_i\dot{q}_j - \frac{1}{2}\sum_{ij} K_{ij} q_i q_j$

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i} \rightarrow \frac{d}{dt}\frac{1}{2}\sum_j M_{ij}\dot{q}_j = \frac{1}{2}\sum_j M_{ij}\ddot{q}_j = -\frac{1}{2}\sum_j K_{ij} q_j$$

$$M\ddot{q} = -Kq \qquad q = \mathrm{Re}\, z(t)$$

$$(K - \omega^2 M)\, a = 0 \leftarrow z(t) = a e^{i\omega t}$$

L17 Tripple coupled Pendulum.



$$T = \frac{1}{2} m (V_1^2 + V_2^2 + V_3^2) = \frac{1}{2} m L^2 (\dot\varphi_1^2 + \dot\varphi_2^2 + \dot\varphi_3^2)$$

$$U_g = \frac{1}{2} mg L (\varphi_1^2 + \varphi_2^2 + \varphi_3^2)$$

used $(1 - \cos\varphi) \approx \varphi^2/2$

$$U_K = \frac{1}{2} k L^2 (\sin\varphi_2 - \sin\varphi_1)^2 + \frac{1}{2} k L^2 (\sin\varphi_3 - \sin\varphi_2)^2$$ use $\sin\varphi \approx \varphi$

$$U_k = \frac{1}{2} k L^2 \left((\varphi_2 - \varphi_1)^2 + (\varphi_3 - \varphi_2)^2 \right)$$

natural units: lets assume that mass is measured in units of m $m \to 1$, same for L

$$\mathcal{L} = \frac{1}{2}(\dot\varphi_1^2 + \dot\varphi_2^2 + \dot\varphi_3^2) - \frac{1}{2} g (\varphi_1^2 + \varphi_2^2 + \varphi_3^2) - \frac{1}{2} k \left[(\varphi_2 - \varphi_1)^2 + (\varphi_3 - \varphi_2)^2 \right]$$

$$M \ddot\varphi = - K \varphi \qquad M = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} \quad K = \begin{vmatrix} g+k & -k & 0 \\ -k & g+2k & -k \\ 0 & -k & g+k \end{vmatrix}$$

$$\det(K - \omega^2 M) = (g - \omega^2)(g + k - \omega^2)(g + 3k - \omega^2)$$

$$\omega_1^2 = g \qquad \omega_2^2 = g + k \qquad \omega_3 = g + 3k$$

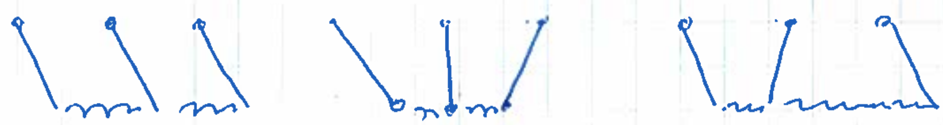
How to restore normal units? g/ω_1^2 — m — length. which is measured in units L

$$g/\omega_1^2 = L \qquad \omega_1 = \sqrt{g/L}$$

$\omega^2 = k \to k/\omega^2$ — mass, which is measured in m

$$\omega_2 = \sqrt{\frac{g}{L} + \frac{k}{m}} \qquad \omega_3 = \sqrt{g/L + 3k/m}$$

modes



$\varphi_1 = \varphi_2 = \varphi_3$ $\varphi_2 = 0$, $\varphi_1 = -\varphi_3$ $\varphi_3 = \varphi_1$ $\varphi_2 = -2\varphi_1$