L18 Hamiltonian Mechanics

- $\mathcal{L} = \mathcal{L}(\vec{q}, \dot{\vec{q}}) = T - U$ $\quad \vec{q} = q_1 \dots q_N$, $\dot{\vec{q}} = \dot{q}_1 \dots \dot{q}_N$

$$\frac{\partial \mathcal{L}}{\partial q_i} = \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{d}{dt} p_i = \dot{p}_i$$

q - configuration space
$q, \dot{q}$ - point in state space

$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ - generalized momentum (conjugate to q_i)
(canonical)

$$\frac{d\mathcal{L}(q\dot{q})}{\partial t} = \sum_i \frac{\partial \mathcal{L}}{\partial q_i}\dot{q}_i + \sum_i \frac{\partial \mathcal{L}}{\partial \dot{q}_i}\ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t} = \sum \dot{p}_i \dot{q}_i + p_i \ddot{q}_i + \frac{\partial \mathcal{L}}{\partial t}.$$

$$\frac{d\mathcal{L}}{dt}(q\dot{q}) = \frac{d}{dt}\sum_i p_i \dot{q}_i + \frac{\partial \mathcal{L}}{\partial t} \Rightarrow \frac{\partial \mathcal{L}}{\partial t} = -\frac{d}{dt}\left(\sum p_i \dot{q}_i - \mathcal{L}\right);$$

$$H = \sum_i p_i \dot{q}_i - \mathcal{L} = \text{const if } \frac{\partial \mathcal{L}}{\partial t} = 0$$

- natural generalized coordinates (section 7.8)
$\vec{r}_\alpha = \vec{r}_\alpha(\vec{q})$ — no explicit dependence on time.
$\sum p_i \dot{q}_i = 2T$
- than H is total energy of the system $H = T + U$
- instead of $\vec{q}, \dot{\vec{q}}$ (state space) use $(\vec{q}\,\vec{p})$ - phase space
Hamiltonian equations determine a unique path in the phase space.
- Hamiltonian equations (1D case)

$\mathcal{L}(q\dot{q}) = T(q, \dot{q}) - U(q)$ $\quad$ pendulum: $\frac{1}{2}mL^2\dot{\varphi}^2 - mgL\varphi^2/2$

$\mathcal{L} = \frac{1}{2}A(q)\dot{q}^2 - U(q)$ $\quad p = \frac{\partial \mathcal{L}}{\partial \dot{q}} = A(q)\dot{q} \rightarrow \dot{q} = \frac{p}{A(q)}$

$H = p\dot{q} - \mathcal{L} = 2T - T + U = T + U \Rightarrow$

$H = p\,\dot{q}(p, q) - \mathcal{L}(q, \dot{q}(p, q))$

L18 Hamilton's equations

$$H(p,q) = p\dot{q}(p q) - \mathcal{L}(q,\dot{q}(p,q))$$

$$q: \quad \frac{\partial H}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial q} = p\frac{\partial \dot{q}}{\partial q} - \frac{\partial \mathcal{L}}{\partial q} - p\frac{\partial \dot{q}}{\partial q} = -\dot{p}$$

$$p: \quad \frac{\partial H}{\partial q} = \dot{q} + p\frac{\partial \dot{q}}{\partial p} - \frac{\partial \mathcal{L}}{\partial \dot{q}}\frac{\partial \dot{q}}{\partial p} = \dot{q}$$

$$\frac{\partial H}{\partial q} = -\dot{p} \qquad \frac{\partial H}{\partial p} = \dot{q}$$

- 1D motion

$$\mathcal{L}: \quad \mathcal{L}(x\,\dot{x}) = \frac{1}{2}m\dot{x}^2 - U(x)$$

$$H: \quad H(xp) = \frac{1}{2}\frac{p^2}{m} + U(x) \qquad \text{use } p = \frac{\partial \mathcal{L}}{\partial \dot{x}} = m\dot{x}$$

$$\frac{\partial H}{\partial p} = \frac{p}{m} = \dot{x} \qquad \frac{\partial H}{\partial q} = +\frac{\partial U}{\partial x} = -\dot{p} \quad \text{- Newton's Law}$$

- oscillator:

$$\mathcal{L} = \frac{1}{2}m\dot{x}^2 - \frac{1}{2}kx^2$$

$$H = \frac{1}{2}\frac{p^2}{m} + \frac{1}{2}kx^2 = \frac{1}{2}\mathbb{P}^2 + \frac{1}{2}Q^2 \qquad \mathbb{P} = \frac{p}{\sqrt{m}} \quad Q = \sqrt{k}\,x.$$

$$\left\{ \frac{p}{m} = \dot{x} \qquad kx = \dot{p} \right\}$$