L20 Canonical transformations

$\dot{q}_i = \frac{\partial H}{\partial p_i} = f_i(\vec{p},\vec{q}) \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} = g_i(\vec{p},\vec{q})$

$z = (p,q)$ – point in the phase space

evolution of z – phase space path – is defined by the equations $\dot{\vec{z}} = \vec{h}(z) \qquad h = (f_1 \ldots f_n, g_1 \ldots g_n)$

Transformations

agrangian: $\vec{Q}(\vec{q}) \quad q \to \vec{Q}(\vec{q})$ new coordinates $\vec{Q}$

amiltonian: $(p,q) \to (Q,\mathbb{P}) \quad Q = Q(p,q) \quad \mathbb{P} = \mathbb{P}(p,q)$

nonical transformations:

ew set $(\mathbb{P},Q)$ satisfies Hamilton's equations.

$\frac{\partial H}{\partial Q} = -\dot{\mathbb{P}} \qquad \frac{\partial H}{\partial p} = \dot{Q} \quad$ so $\quad \frac{\partial \dot{p}}{\partial p} = -\frac{\partial \dot{Q}}{\partial Q} = \frac{\partial^2 H}{\partial p \partial Q}$

such transformation preserves the "phase volume"

ase-space orbits: $z(t)$

ince $\dot{z}(t) = h(z(t))$ only one orbit can go through a point $z_0 = z(t_0)$

if H does not depend on t explicitely,

only one (orbit) path can cross z_0 at any time

ample 13.5 Harmonic oscillator.

$H = T + U = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \qquad U = \frac{1}{2}kx^2 \quad (\frac{k}{m} = \omega^2) \to \frac{1}{2}m\omega^2 x^2$

$\dot{x} = \frac{p}{m} \qquad \dot{p} = -m\omega^2 x \qquad x = A\cos(\omega t - \delta) \qquad p = -m\omega A \sin(\omega t - \delta)$

total energy $= \frac{1}{2}m\omega^2 A^2 = \frac{p^2}{2m} + \frac{1}{2}m\omega^2 x^2 \Rightarrow \frac{1}{A^2}\left(x^2 + \left(\frac{p}{m\omega}\right)^2\right) = 1$

$x^2 + \left(\frac{p}{m\omega}\right)^2 = A^2 = x^2 + \mathbb{P}^2$

different initial condit…

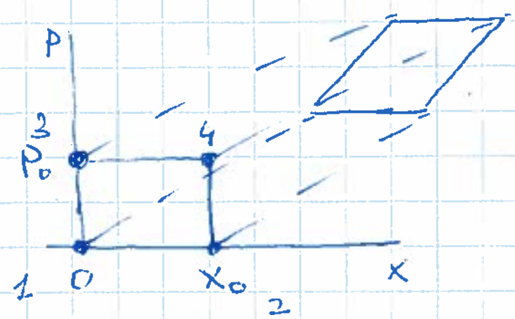
L20; Example 13.6 Falling Body.

$$H = T + U = \frac{1}{2}m\dot{x}^2 - mgx = \frac{p^2}{2m} - mgx$$

$$\dot{x} = \frac{\partial H}{\partial p} = \frac{p}{m} \qquad \dot{p} = -\frac{\partial H}{\partial x} = mg$$

$$x = x_0 + \frac{p_0}{m}t + g t^2/2 \qquad p = p_0 + mgt$$

Consider 4 initial conditions.



$P_1(t) - P_0(t) = P_d(0) - P_c(0) = \text{const}$

$X(t) - X_0(t) = X_d(0) - X_0(0) = \text{const}$

$X_0 P_0$ = phase volume which is conserved.