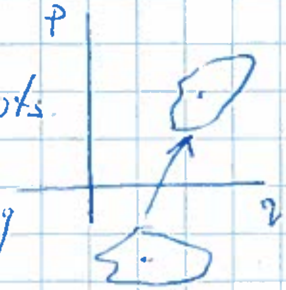
L21 Liouville's Theorem

Volume in phase space occupied by "fluid" of space ph. dots given a state $z=(p q)$ it evolves into a state $z'=(p'q')$ according to Hamilton's equation

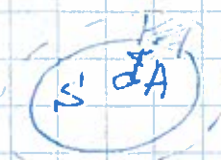


$\dot{z}(t) = H(z)$ $\quad \dot{z}$ - phase space velocity.

each dot in phase space moves with it $\dot{z}$
the volume occupied by phase space dots may change shape, but the volume can not change ← Liouville Theorem

- changing volumes (use analogy of a 3D fluid)
  each point $\vec{r}$ moves with velocity $\vec{u}(\vec{r})$

$\vec{u} = \dot{\vec{r}} = f(\vec{r}) \rightarrow$ totally equivalent to $\dot{\vec{z}} = H(z)$



$\delta V = \hat{n}\,\vec{u}\,\delta t\, dA$ $\qquad \hat{n}$ - unit vector $\perp$ to S

$\hat{n}\,\vec{u}\,\delta t = \delta h$

δA - small area of S

$$\Delta V = \int_S \hat{n}\,\vec{u}\,\delta t\, dA \rightarrow \frac{\Delta V}{\delta t} = \frac{dV}{dt} = \int \hat{n}\,\vec{u}\, dA$$

rate of V change

- $\hat{n}\,\vec{u} > 0$ V is expanding
- $\hat{n}\,\vec{u} < 0$ V is decreasing
- incompressible fluid $\quad V = const \rightarrow \int \hat{n}\,\vec{u}\, dA = 0$

- The Divergence Theorem (Gauss Theorem)
  * divergence - vector operator $\vec{\nabla}$

$$\vec{\nabla}\cdot\vec{u} = \frac{\partial u_x}{\partial x} + \frac{\partial u_y}{\partial y} + \frac{\partial u_z}{\partial z} = \sum_i \frac{\partial u_i}{\partial q_i} \Rightarrow$$

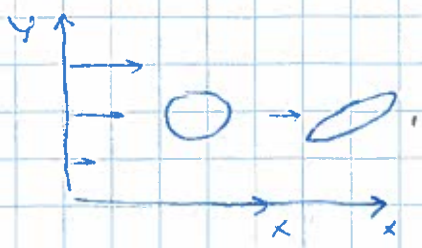
$\vec{u}$ could be any vector

replace surface integral with Volume Integral

$$\int_S \hat{n}\,\vec{u}\, dA = \int_V \vec{\nabla}\cdot\vec{u}\, dV \quad \text{- Divergence Theorem}$$

L21

Example 13.7 A Shearing Flow.



$\vec{u} = ky\hat{x} = \{ky, 0, 0\}$

$\vec{\nabla}\cdot\vec{u} = \left(\frac{\partial ky}{\partial x} = 0, 0, 0\right) = 0$

$\frac{dV}{dt} = \int_V \vec{\nabla}\vec{u}\, dV = 0$

consider a small volume $V \to \vec{\nabla}\cdot\vec{u} = \text{const}$

$\frac{1}{V}\frac{dV}{dt} = \vec{\nabla}\vec{u}$ outward flow per volume.

Liouville's Theorem: $\quad z = (\vec{q}, \vec{p})$

$u = \dot{z} = (\dot{\vec{q}}, \dot{\vec{p}}) = \left(\frac{\partial H}{\partial p_i}, -\frac{\partial H}{\partial q_i}\right)$ - "velocity" in phase space.

$\frac{dV}{dt} = \int_V \vec{\nabla}\vec{u}\, dV$

$\vec{\nabla}\vec{u} = \sum_i \frac{\partial \dot{q}_i}{\partial q_i} + \frac{\partial \dot{p}_i}{\partial p_i} = \sum_i \frac{\partial}{\partial q_i}\left(\frac{\partial H}{\partial p_i}\right) + \frac{\partial}{\partial p_i}\left(-\frac{\partial H}{\partial q_i}\right) = 0$