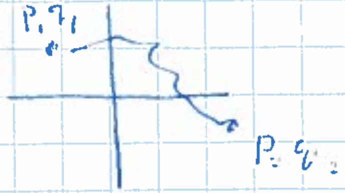
Action.

L22

$$S = \int \mathcal{L}(q, \dot{q})\, dt = \int dt \left[\sum p_i \dot{q}_i - H(p, q) \right]$$



in $\mathcal{L}$ we vary q

in H we vary p & q independently

$$p \to p + \delta p \qquad q \to q + \delta q$$

$$\delta S = \int dt \left\{ \sum_i \delta p_i \dot{q}_i + p_i \delta \dot{q}_i - \frac{\partial H}{\partial p_i} \delta p_i - \frac{\partial H}{\partial q_i} \delta q_i \right\}$$

$$= \int dt \left\{ \sum_i \left(\dot{q}_i - \frac{\partial H}{\partial p_i} \right) \delta p_i + \left(-\dot{p}_i - \frac{\partial H}{\partial q} \right) \delta q_i \right\}$$

use $\int_1^2 dt\, p_i \delta \dot{q}_i = -\int_1^2 dt\, \dot{p}_i \delta q_i + p_i \delta q_i \Big|_1^2 \to 0$

$\delta S = 0$ when
$$\dot{q}_i - \frac{\partial H}{\partial p_i} = 0$$
$$\dot{p}_i + \frac{\partial H}{\partial q_i} = 0$$
} Hamilton's equations

IN HM $\vec{p}_i$ $\vec{q}$ are treated on equal footing

- Free particle

$\mathcal{L}$: $x(t)$ $\quad p = m\dot{x}$

H: $x(t)$ $p(t)$ independent functions

$p = m\dot{x}$ is one of equation of motion followed from H's equations

$$H = \frac{p^2}{2m} \qquad \dot{x} = p/m \qquad \dot{p} = 0$$

Poisson formulation of CM

L22

$F(q,p)$ – some function, we are interested how it changes with time

$$\frac{dF(q,p)}{dt} = \sum_i \frac{\partial F}{\partial q_i}\dot{q}_i + \sum \frac{\partial F}{\partial p_i}\dot{p}_i + \frac{\partial F}{\partial t} =$$

$$= \sum_i \frac{\partial F}{\partial q_i}\frac{\partial H}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial H}{\partial q_i} + \frac{\partial F}{\partial t} = \{F,H\} + \frac{\partial F}{\partial t}$$

if F is not explicit function of $t \rightarrow \frac{\partial F}{\partial t} = 0$

so $\boxed{\dot{F} = \{F,H\}}$ ← the most compact formulation of CM

$$\{F,G\} = \sum_i \frac{\partial F}{\partial q_i}\frac{\partial G}{\partial p_i} - \frac{\partial F}{\partial p_i}\frac{\partial G}{\partial q_i}$$ – Poisson bracket.

for any 2 arbitrary functions of p & q $F(q,p)$ & $G(q,p)$ however $\{F,G\} \neq \dot{F}$ this is true if only $G = H$.

- Harmonic oscillator by using Algebra of Poisson Bracket (next page)

$H = \frac{p^2}{2m} + \frac{k}{2}q^2$ lets calculate equation of motion using only PB.

$$\dot{q} = \{q,H\} = \{q, \frac{p^2}{2m}\} = \frac{1}{2m}p\{q,p\} + \frac{1}{2m}\{q,p\}p = \frac{p}{m}$$

$$\dot{p} = \{p,H\} = \{p, \frac{k}{2}q^2\} = \frac{k}{2}2q\{p,q\} = -kq$$

* $\dot{F} = \{F,H\}$ if H is time translation invariant (does not depend on t explicitely) the Poisson Bracket gives the rate of change of F in time

if $\{F,H\} = 0$ $\dot{F} = 0$ conserved.

L22 # Poisson Brackets Algebra

A formulation of classical mechanics that resembles QM (in fact it is opposit)

Consider 2 points in phase space

$f(p_i q_i)$ $g(p,q)$ define.

$$\{f,g\} = \sum_i \left\{ \frac{\partial f}{\partial q_i}\frac{\partial g}{\partial p_i} - \frac{\partial f}{\partial p_i}\frac{\partial g}{\partial q_i} \right\}$$ – Poisson bracket

- $\{f g\} = -\{g f\}$ antisymmetric
- $\{\alpha f + \beta h, g\} = \alpha\{f,g\} + \beta\{g,h\}$ – linear.
- $\{f h, g\} = f\{h,g\} + \{f,g\}h$ – Leibniz rule.
- $\{f\{g,h\}\} + \{g\{h,f\}\} + \{h,\{f,g\}\} = 0$

Jacobi identity (not obvious, but can be proved easily)

these identities are all obeyed by the commutator of matrices $[F,G] = FG - GF$

* $\{q_i, q_j\} = 0 = \{p_i, p_j\}$
* $\{q_i\ p_j\} = \delta_{ij}$

L 22 Symmetry.

* t - invariance $\rightarrow H = H(q, p)$ $\{H, H\} = 0$ $\dot{H} = 0$
conservation of energy.

* q - invariance $\rightarrow H = H(p)$ $\{p, H(p)\} = 0$ $\dot{p} = 0$
conservation of momentum

Noether's theorem.

* Suppose that I & J are conserved.

$\{I, H\} = 0$ $\{J, H\} = 0$ what is $\{\{I, J\}, H\} = ?$

use Jacobi rule $\rightarrow$ $\{\{I, J\} H\} = 0$

so $\{I, J\}$ is also conserved

The poisson bracket of 2 constants of motions is also a constant of motion

The conserved quantities form "an algebra"

- Angular momentum $\vec{\ell} = \vec{r} \times \vec{p}$

$\ell_x = y p_z - z p_y$

$\ell_y = z p_x - x p_z$

$\ell_z = x p_y - y p_x$

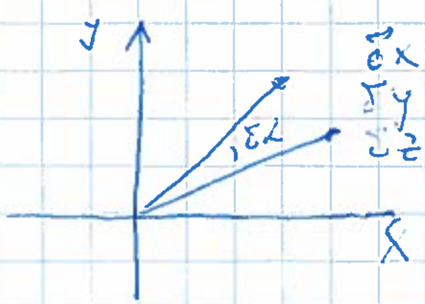
particle in central field
ℓ is conserved.

$\{x, \ell_z\} = \{x, x p_y - y p_x\} = -y\{x, p_x\}$

$\{y, \ell_z\} = \{y, x p_y - y p_x\} = x\{y, p_y\} = x$

$\{z, \ell_z\} = \{z, x p_y - y p_x\} = 0$

H is invariant wrt rotation.
ℓ is conserved.



$\delta x = -y\,\delta\alpha \rightarrow \frac{\partial x}{\partial \alpha} = -y$

$\delta y = x\,\delta\alpha \rightarrow \frac{\partial y}{\partial \alpha} = x$

$\delta z = 0 \rightarrow \frac{\partial z}{\partial \alpha} = 0$

$$\boxed{\{\vec{r}, \ell_z\} = \frac{\partial \vec{r}}{\partial \alpha}}$$

$\{\vec{p}, \ell_z\} = \frac{\partial \vec{p}}{\partial \alpha}$

prove it