# Continuum Mechanics

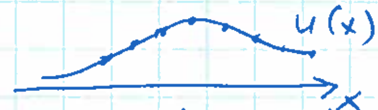
L32

a) mechanics of point masses
b) mechanics of rigid bodies
c) Continuum mechanics.

discrete to continuum mechanics
diff. equations to partial diff. eq.

→ waves.

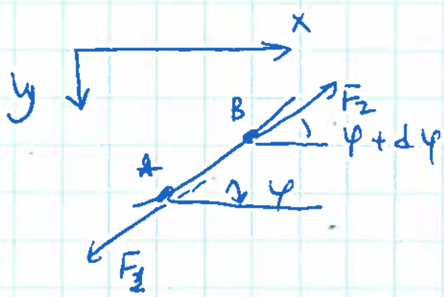
Motion of a string:




can approximate first with a discrete system.

masses $m_1 \dots m_n$ at displacements $u_1 \dots u_n$

for $n \to \infty$ $m \to \mu \cdot (x)$ density, $u \to u(x)$ or $u(x,t)$ displacement.




$\vec{F}_{net} = \vec{F}_1 + \vec{F}_2$ ← tension forces.

$F_{1x} = T\cos\varphi$ $F_{2x} = T\cos(\varphi + d\varphi)$

$F_{xnet} = T\cos(\varphi + d\varphi) - T\cos\varphi \to 0$ for small $\varphi$.

all motion is in y direction

$$F_{ynet} = T\sin(\varphi + d\varphi) - T\sin\varphi \approx T\cos\varphi \, d\varphi = T\cos\varphi \frac{\partial \varphi}{dx} dx$$

$$F_{ynet} = T \frac{\partial \varphi}{dx} dx \to T \frac{\partial^2 u}{\partial x^2} dx$$ use $\varphi = \frac{\partial u}{dx}$ - slope.

$\frac{\partial u}{\partial t}$ - velocity $\frac{\partial^2 u}{\partial t^2}$ - acceleration of a segment of string with mass $\Delta m = \mu dx$

So $F_{ynet} = \mu \frac{\partial^2 u}{dt^2} dx = T \frac{\partial^2 u}{dx^2} dx$ or.

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad c^2 = T/\mu$$

1D wave equation.

L32 The Wave Equation

the general solution

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \longrightarrow -4c^2 \frac{\partial}{\partial \xi}\frac{\partial u}{\partial \eta} \qquad \xi = x - ct \quad \eta = x + ct$$

$u(\xi, \eta)$ ; use $\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = -c\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(\frac{\partial u}{\partial \xi} - \frac{\partial u}{\partial \eta}\right)(-c) = c^2\left[\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta}\right]$$

$$c^2\frac{\partial^2 u}{\partial x^2} = c^2 \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) = c^2\left[\frac{\partial^2 u}{\partial \xi^2} + \frac{\partial^2 u}{\partial \eta^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta}\right]$$

$$\frac{d^2 u}{d t^2} - c^2\frac{\partial^2 u}{\partial x^2} = -4c^2\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 = \frac{\partial}{\partial \xi}\left(\frac{\partial u}{\partial \eta}\right)$$

$$\frac{\partial u}{\partial \eta} = h(\eta) \qquad u = \underbrace{\int h(\eta)\, d\eta}_{\text{"} g(\eta)} + \underbrace{f(\xi)}_{\text{"constant' of } \eta}$$

$u = g(\eta) + f(\xi)$ — general solution.

$$u(x,t) = g(x+ct) + f(x-ct)$$

first consider $u(x,t) = f(x-ct)$

$t = 0$ : $u(x,0) = f(x)$ $\quad$ $f(x)$ shifted by $X_0$

$t = t_0$ ; $u(x,t_0) = u(x') = f(x')$ $\quad X = x' + X_0 \quad X_0 = ct$

$f(x)$ travels to the left with velocity c

$u = g(x+ct) \longrightarrow$ wave that travels to the right.

L32 Triangular Waves textbook. Example 16.1

initial condition


release string at $t=0$ $\quad u(x,0)=u_0(x)$

$u(x,t) = f(x-ct) + g(x+ct)$

$$\frac{\partial u}{\partial t} = -c\frac{\partial f}{\partial \xi} + c\frac{\partial g}{\partial \eta} = 0 \text{ at } t=0$$

$$\left.\frac{\partial f}{\partial \xi}\right|_{t=0} = \left.\frac{\partial g}{\partial \eta}\right|_{t=0} \longrightarrow f(\xi)_{t=0} = g(\eta)_{t=0} = f(x) = g(x)$$

but $u_0(x) = f(x)+g(x) \rightarrow f(x) = g(x) = \frac{1}{2}u_0(x)$

at any timet $t$

$$u(x,t) = f(x-ct) + g(x+ct) = \tfrac{1}{2}u_0(x-ct) + \tfrac{1}{2}u_0(x+ct)$$

The initial disturbance falls into 2 triangular waves traveling left & right.

Harmonic waves: $f, g$ = sin & cos

for example $u(x,t) = A\sin(k(x-ct))$

$A$, $k$ arbitrary constants.

$u(x,t) = A\sin(kx-\omega t)$ $\quad \omega = kc$ angular frequency

$k$ – wave number $= 2\pi/\lambda$ $\quad \lambda$ – wavelength

Superposition of 2 waves:

$$u(x,t) = A\sin(kx-\omega t) + A\sin(kx+\omega t) =$$
$$= 2A\sin(kx)\cos\omega t \quad \text{– standing wave.}$$