PH4222, Section 3801, Spring 2014, Homework 7

Due at the start of class on Friday, March 14. Half credit will be available for late homework submitted no later than the start of class on Monday, March 17.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. A massless spring (force constant k_1) is suspended from the ceiling, with a mass m_1 hanging from its lower end. A second spring (force constant k_2) is suspended from m_1, and a second mass m_2 is suspended from the second spring's lower end.

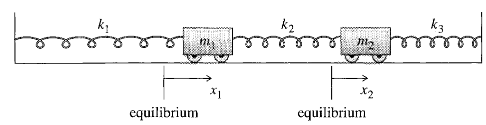
   a) Use coordinates y_1 and y_2 measured from the masses' equilibrium positions (and ignoring gravity), write down the Lagrangian for the system.

   b) Show that the equations of motion can be written in the matrix form $\mathbf{M}\ddot{\mathbf{y}} = -\mathbf{K}\mathbf{y}$, where $\mathbf{y}$ is the 2 x 1 column made up of y_1 and y_2.

   c) By a suitable substitution, solve the eigenvalue problem and show that ω^2 for this problem is always real, and always positive.

   d) By including gravity, AND using the full extension of the springs from their un-extended lengths, show that your Lagrangian in a) is justified.

2. In discussing the two carts shown, we ignored any difference between the equilibrium lengths L_1, L_2, L_3 of the three springs and their natural, un-stretched lengths l_1, l_2, l_3. However, this simplification is not needed, and the three springs could all be in tension (or compression) at the equilibrium position.



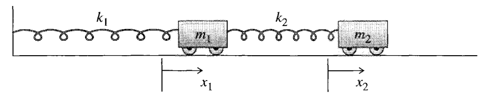
   a) Find the relations among these six lengths (and the three spring constants k_1, k_2, k_3) required for the two carts to be in equilibrium.

   b) Letting x_1 and x_2 be the displacements from equilibrium, write down the full Lagrangian.

   c) Simplify your Lagrangian by using your results from a).

   d) Hence find the equation of motion for each cart and show that it is independent of how L_1, L_2, L_3 compare with l_1, l_2, l_3, just as long as x_1 and x_2 are measured from the carts' equilibrium positions.

3. Consider the system shown, where x_1 and x_2 are the displacements from equilibrium of the two carts.



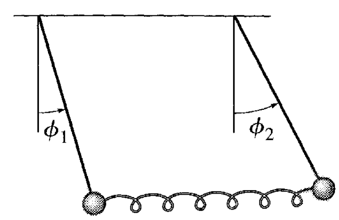
a) Write down the Lagrangian for the system.

b) Find the equations of motion.

c) Find the normal frequencies, ω_1 and ω_2, for the two carts, assuming that $m_1 = m_2$ and $k_1 = k_2$.

d) Find and describe the motion for each of the normal modes in turn.

4. Consider two identical plane pendulums (each of length L and mass m) that are joined by a massless spring (force constant k) as shown. The pendulums' positions are specified by the angles ϕ_1 and ϕ_2 shown. The natural length of the spring is equal to the distance between the two supports, so the equilibrium position is at $\phi_1 = \phi_2 = 0$ with the two pendulums vertical.



a) Write down the total kinetic energy and the gravitational and spring potential energies. [Assume that both angles remain small at all times. This means that the extension of the spring is well approximated by $L(\phi_1 - \phi_2)$.]

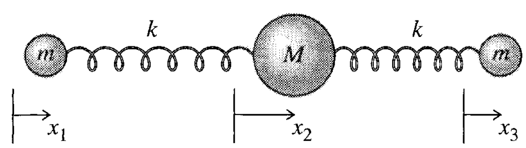
b) Write down the Lagrange equations of motion, keeping all terms up to quadratic order in the angles and their derivatives.

c) Find and describe the normal modes for these two coupled pendulums.

d) What would be a natural choice for the normal coordinates ξ_1 and ξ_2?

e) Show that even if both pendulums are subject to a resisitive force of magnitude bv (with b small), the equations of motion for ξ_1 and ξ_2 are still uncoupled.

5. As a model of a linear triatomic molecule (such as CO_2), consider a system with two identical atoms of mass m connected by two identical springs to a single atom of mass M. To simplify matters, assume that the system is confined to move in one dimension.



a) Write down the Lagrangian and find the normal frequencies of the system. Show that one of the normal frequencies is zero.

b) Find and describe the motion in the normal modes whose frequencies are nonzero.

c) Do the same for the mode with zero frequency. [*Hint*: This requires some thought. It isn't immediately clear what oscillations of zero frequency are. Notice that the eigenvalue equation $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{a} = 0$ reduces to $\mathbf{Ka} = 0$ in this case. Consider a solution $\mathbf{x}(t) = \mathbf{a}f(t)$, where $f(t)$ is an undetermined function of t and use the equation of motion, $\mathbf{M}\ddot{\mathbf{x}} = -\mathbf{Kx}$, to show that this solution represents motion of the whole system with constant velocity. Explain why this kind of motion is possible here but not in the previous examples.]