

PH4222, Section 3801, Spring 2015, Homework 4

Due at the start of class on Friday, February 6. Half credit will be available for late homework submitted no later than the start of class on Monday, February 9.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. Consider the solution to the Kepler orbit problem given by:

$$r = \frac{c}{(1 + \epsilon \cos \phi)}.$$

- a) In the case that $0 < \epsilon < 1$, show that the equation in cartesian coordinates can be rewritten as:

$$\frac{(x + d)^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and find a , b and d in terms of ϵ and c .

- b) In the case that $\epsilon = 1$, show that the equation can be rewritten as a parabola in cartesian coordinates, and find the intercepts on the x and y -axes.

- c) In the case that $\epsilon > 1$, show that the equation in cartesian coordinates can be rewritten as:

$$\frac{(x + d)^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and find a , b and d in terms of ϵ and c .

2. Two particles of equal masses $m_1 = m_2$ move on a frictionless horizontal surface in the vicinity of a fixed force center, with potential energies $U_1 = \frac{1}{2}kr_1^2$ and $U_2 = \frac{1}{2}kr_2^2$. In addition, they interact with each other via a potential energy $U_{12} = \frac{1}{2}\alpha kr^2$, where r is the distance between them and α and k are positive constants.

- a) Find the Lagrangian in terms of the CM position \mathbf{R} and the relative position $\mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2$.

- b) Write down and solve the Lagrange equations for the CM and relative coordinates X , Y and x , y . Describe the motion.

3. Consider the effective potential energy:

$$U_{\text{eff}}(r) = -\frac{\gamma}{r} + \frac{l^2}{2\mu r^2},$$

Where γ represents the strength of the attractive force between the Sun and a planet (or comet), l is the angular momentum of the system and μ is the reduced mass.

- Write down an equation representing the conservation of energy for the reduced system.
 - Hence, find the radius at which a planet (or comet) with system angular momentum l can orbit the Sun in a circular orbit with fixed radius. *Hint:* Look at dU_{eff}/dr .
 - Show that this circular orbit is stable, in the sense that a small radial nudge will cause only small radial oscillations. *Hint:* It may help to differentiate your result from a) to obtain an equation for the acceleration, and express it in terms of a small amplitude displacement away from the equilibrium orbit.
 - Show that the period of these oscillations is equal to the planets orbital period.
4. A particle of mass m moves with angular momentum l in the field of a fixed force center with $F(r) = -\gamma/r^2 + \lambda/r^3$, where γ and λ are positive.

a) Write down the transformed radial equation (*i.e.*, $d^2u/d\phi^2 = \dots$, where $u = 1/r$).

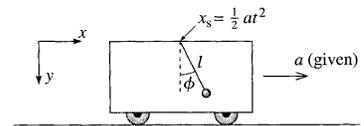
b) Prove that the orbit has the form:

$$r(\phi) = \frac{c}{1 + \epsilon \cos(\beta\phi)}$$

where c , β and ϵ are positive constants.

- Find c and β in terms of the given parameters, and describe the orbit for the case that $0 < \epsilon < 1$.
- For what values of β is the orbit closed?
- What happens to your results as $\lambda \rightarrow 0$?

5. Consider a railroad car which has a pendulum suspended from its ceiling and is being forced to accelerate with a fixed acceleration a .



It is natural to specify the position of the pendulum by the angle ϕ as usual, but recognize that, in the first instance, this gives the pendulum's position relative to the accelerating, and hence non-inertial, reference frame of the car.

- Specify the bob's position (x, y) relative to an inertial frame, in terms of ϕ and t .
- Hence write down the Lagrangian and the Euler-Lagrange equation for the bob.
- Show that the bob can have stable angle of suspension for non-zero a , and find it.
- What is the frequency of small oscillations about this equilibrium position?