

PH4222, Section 3801, Spring 2015, Homework 8

Due at the start of class on Friday, March 20. Half credit will be available for late homework submitted no later than the start of class on Monday, March 23.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. For a linear molecule like CO_2 , the eigenvectors for longitudinal motion are:

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \hat{\mathbf{e}}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \text{and} \quad \hat{\mathbf{e}}_3 = \frac{1}{\sqrt{2(1+2\gamma^2)}} \begin{pmatrix} 1 \\ -2\gamma \\ 1 \end{pmatrix},$$

where $\gamma = M_{\text{O}}/M_{\text{C}} \neq 1$.

- By taking inner products, show that these are not all orthogonal. How many of the inner products are not zero?
- Show that: $\xi_1 \hat{\mathbf{e}}_1 + \xi_2 \hat{\mathbf{e}}_2 + \xi_3 \hat{\mathbf{e}}_3 = \mathbf{0} \Rightarrow \xi_i = 0 \forall 1 \leq i \leq 3$, *i.e.*, that the $\hat{\mathbf{e}}_i$ are linearly independent, even though they are not all orthogonal.
- Let x , y and z represent the displacements of the three atoms from some equilibrium. Suppose:

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \xi_1 \hat{\mathbf{e}}_1 + \xi_2 \hat{\mathbf{e}}_2 + \xi_3 \hat{\mathbf{e}}_3$$

and find the ξ_i (the normal coordinates for the problem) in terms of x , y , and z .

- Let $\hat{\mathbf{e}}_4 = \alpha \hat{\mathbf{e}}_1 + \beta \hat{\mathbf{e}}_3$ be orthogonal to, and linearly independent of, $\hat{\mathbf{e}}_1$ and $\hat{\mathbf{e}}_2$. Find α and β .
2. Consider a generic Lagrangian for a 1-D system with generalized coordinate q given by:

$$\mathcal{L} = \frac{1}{2} f(q) \dot{q}^2 - U(q).$$

- Find p , the momentum canonical to q and use your result to express $\dot{q} = \dot{q}(q, p)$.
- Eliminate \dot{q} and write down the Hamiltonian $\mathcal{H} = \mathcal{H}(q, p) = p\dot{q}(q, p) - \mathcal{L}(q, \dot{q}(q, p))$.

Answer the parts below entirely by using the formal relation $\mathcal{H}(q, p) = p\dot{q}(q, p) - \mathcal{L}(q, \dot{q}(q, p))$.

- Find $\partial\mathcal{H}/\partial p$ and show that the coefficient of $\partial\dot{q}/\partial p$ is exactly zero, so that the final result is just \dot{q} .
- Find $\partial\mathcal{H}/\partial q$ and show that the coefficient of $\partial\dot{q}/\partial q$ is exactly zero. Then by using the EL equations for q , show that the final result is just $-\dot{p}$.

3. In problem 1.3 we considered two particles, of mass m_1 and m_2 respectively, confined to the x -axis and connected by a spring with potential energy $U = \frac{1}{2}kx^2$, where $x = x_1 - x_2 - l$ is the extension of the spring and l is its un-stretched length. There we found the Lagrangian for the reduced mass motion was:

$$\mathcal{L} = \frac{1}{2}\mu\dot{x}^2 - \frac{1}{2}kx^2.$$

- a) Let q , the generalized coordinate, be x . Find its canonical momentum p , and solve for \dot{q} as a function of q and p .
 - b) Write down the Hamiltonian \mathcal{H} for the reduced system in terms of q and p alone.
 - c) Write down Hamiltonian's equations of motion for the reduced system.
 - d) These can be combined in the form $\dot{\eta} = -i\omega\eta$, where $\eta = x + i\gamma p$ for some γ and ω both positive. Find γ and ω .
 - e) What is the solution for x and p . Show that this is consistent with your solution for \dot{q} in a).
4. For two masses of (total mass M) moving under the influence of gravity, μ , the reduced mass of the system, is governed by the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\mu \left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) + \frac{G\mu M}{r}.$$

- a) Find the canonical momentum for each of the three coordinates r , θ and ϕ .
 - b) Are any of these conserved and, if so, which?
 - c) Find the Hamiltonian.
 - d) Write down Hamilton's equations of motion for all the variables involved.
5. A bead of mass m is threaded on a frictionless wire that is bent into a helix with cylindrical polar coordinates (ρ, ϕ, z) , satisfying $z = c\phi$ and $\rho = R$, with c and R constants. The z -axis points vertically up and gravity vertically down.
- a) Using ϕ as a generalized coordinate, write down the kinetic and potential energies, and thus the Hamiltonian \mathcal{H} as a function of ϕ and its conjugate momentum p .
 - b) Write down Hamilton's equations and solve for $\ddot{\phi}$ and hence \ddot{z} .
 - c) Explain your result in terms of Newtonian mechanics.

6. The Lagrangian for a non-relativistic particle of mass M and charge Q moving in an external electromagnetic field, can be given by:

$$L(\mathbf{r}, \dot{\mathbf{r}}, t) = \frac{1}{2}M(\dot{x}^2 + \dot{y}^2 + \dot{z}^2) - Q(V - \dot{x}A_x - \dot{y}A_y - \dot{z}A_z),$$

where the scalar V and the components A_i of the vector \mathbf{A} are all functions of (x, y, z, t) .

[*Note:* This is not in the common form: $L(\dot{x}^i, x^i) = T(\dot{x}^i) - U(x^i)$.]

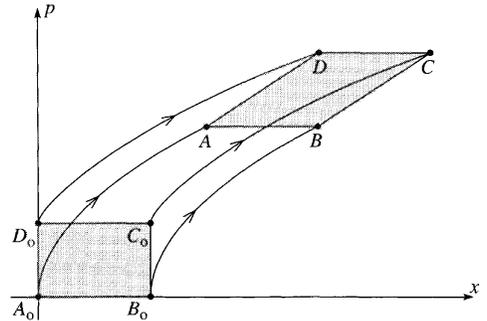
- Let q^i , the generalized coordinates, be x^i . Find their canonical momenta p_i , and solve for \dot{q}^i as a function of q^i and p_i .
- Write down the Hamiltonian $\mathcal{H}(\mathbf{q}, \mathbf{p})$ for this system and show that it is given by:

$$\mathcal{H}(\mathbf{q}, \mathbf{p}) = \frac{(\mathbf{p} - Q\mathbf{A}(\mathbf{q}))^2}{2M} + QV(\mathbf{q})$$

[*Note:* This Hamiltonian plays an important role in the quantum mechanics of charged particles.]

- Write down Hamiltonian's equations of motion for this system.

7. The figure shows phase-space orbits for identical masses in free fall. The points A_o, B_o, C_o and D_o represent the different initial states at $t = 0$, and A, B, C and D the corresponding states at a later time, t .



- Write down the Lagrangian for one mass. Find the canonical momentum and obtain the Hamiltonian.
- Write down and solve Hamiltonian's equations, assuming the initial state in phase space is (q_o, p_o) .
- By appropriate choice of q_o and p_o , write down equations for the four curves shown.
- Calculate the areas of $ABCD$ and $A_oB_oC_oD_o$ and show these areas are the same. [*Note:* This result is an example of Liouville's theorem.]