

PHY4222, Section 3801, Spring 2016, Homework 6

Due at the start of class on Friday, February 26. Half credit will not be available for late homework submitted after Spring break.

Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.

1. Consider a uniform disk rolling at constant speed along a horizontal surface.
 - a) What is the moment of inertia of the disk about its center?
 - b) What is the total kinetic energy of the disk?
 - c) What is the moment of inertia of the disk about its point of instantaneous contact on the horizontal surface?
 - d) What is instantaneous rotational energy of the disk about its point of contact? How does your answer compare with that in b)?
2. A thin, flat, uniform metal triangle lies in the xy plane with its corners at $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 0)$.
 - a) Find the triangle's inertia tensor \mathbf{I} .
 - b) What are its principle moments?
 - c) Find unit vectors in the three directions of the corresponding axes.
 - d) From scratch, recalculate its inertia tensor with the coordinate axes aligned along the principle axes.
3. A rigid body is rotating freely, subject to zero torque.
 - a) Using principle axes, write down its angular momentum \mathbf{L} in terms of the principle moments λ_i and the components of its angular momentum ω as measured in the body frame.
 - b) Write down the relationship between the rates of change of angular momentum as measured in the space and body frames. Relate these to the torque (zero), as measured in the space frame. Hence obtain Euler's Equations for the body.
 - c) For torque-free motion and arbitrary principle moments, show that there is one principle axis about which rotational motion is unstable.

- d) Use Euler's Equations to show that, for torque-free motion, $L^2 = \mathbf{L} \cdot \mathbf{L}$ is constant, even though \mathbf{L} is changing in general. Similarly, show directly from Euler's Equations, that the kinetic energy of rotation is also constant.
- e) If $\lambda_1 = \lambda_2$, show that ω_3 is constant and find the frequency of the free precession of the body's spin axis.
4. Consider a cube, of side a and mass M , rotating about an edge.
- Write down and evaluate an integral for the moment of inertia of the cube about that edge.
 - Suppose the cube starts from rest, balancing on its edge, resting on a horizontal surface, and topples without slipping, onto the surface. What is its kinetic energy at the instant before impact?
 - During its fall, the center of mass moves. Indicate why all the kinetic energy can be considered as rotational energy about the edge which has been at rest on the surface.
 - Find the angular velocity of the cube at the instant of impact.
5. Consider a symmetric top spinning with principle moments $(\lambda_1, \lambda_1, \lambda_3)$ and its tip fixed on a horizontal surface.
- Using Euler angles (ϕ, θ, ψ) , write down, the angular velocity vector $\boldsymbol{\omega}$, the angular momentum \mathbf{L} , and the complete Lagrangian for the top.
 - Find the canonical momenta which are conserved, and write down the equation of motion for the degree of freedom whose canonical momentum is not conserved.
 - By substituting for all first order derivatives in terms of the conserved momenta, show that the equation of motion from b) can be integrated to give an expression for a conserved energy. Identify the effective potential for the problem.
 - Explain what motion occurs if the angle θ_0 at which $\dot{\phi}$ vanishes lies strictly between the turning points (in θ) of the motion.