

**PHY4222, Section 3801, Spring 2017, Homework 8**

**Due at the start of class on Friday, March 17.** Due to the exam next week, half credit will not be available for any homework submitted late.

*Answer all questions. Please write neatly and include your name on the front page of your answers. You must also clearly identify all your collaborators on this assignment. To gain maximum credit you should explain your reasoning and show all working.*

1. For a linear molecule like  $\text{CO}_2$ , the eigenvectors for longitudinal motion are:

$$\hat{\mathbf{e}}_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \hat{\mathbf{e}}_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}, \quad \text{and} \quad \hat{\mathbf{e}}_3 = \frac{1}{\sqrt{2(1+2\gamma^2)}} \begin{pmatrix} 1 \\ -2\gamma \\ 1 \end{pmatrix},$$

where  $\gamma = M_{\text{O}}/M_{\text{C}} \neq 1$ .

- a) By taking inner products, show that these are not all orthogonal. How many of the inner products are not zero?
- b) Show that:  $\xi_1 \hat{\mathbf{e}}_1 + \xi_2 \hat{\mathbf{e}}_2 + \xi_3 \hat{\mathbf{e}}_3 = \mathbf{0} \Rightarrow \xi_i = 0 \forall 1 \leq i \leq 3$ , *i.e.*, that the  $\hat{\mathbf{e}}_i$  are linearly independent, even though they are not all orthogonal.
- c) Let  $x$ ,  $y$  and  $z$  represent the displacements of the three atoms from some equilibrium. Suppose:

$$\mathbf{x} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \xi_1 \hat{\mathbf{e}}_1 + \xi_2 \hat{\mathbf{e}}_2 + \xi_3 \hat{\mathbf{e}}_3$$

and find the  $\xi_i$  (the normal coordinates for the problem) in terms of  $x$ ,  $y$ , and  $z$ .

- d) Let  $\hat{\mathbf{e}}_4 = \alpha \hat{\mathbf{e}}_1 + \beta \hat{\mathbf{e}}_3$  be orthogonal to, and linearly independent of,  $\hat{\mathbf{e}}_1$  and  $\hat{\mathbf{e}}_2$ . Find  $\alpha$  and  $\beta$ .
2. Consider a generic Lagrangian for a 1-D system with generalized coordinate  $q$  given by:

$$\mathcal{L} = \frac{1}{2}f(q)\dot{q}^2 - U(q).$$

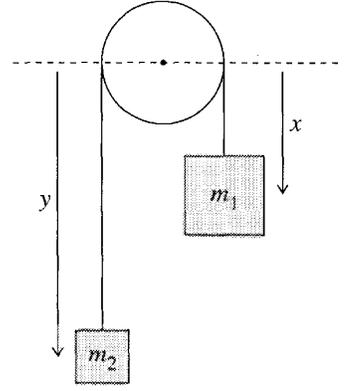
- a) Find  $p$ , the momentum canonical to  $q$  and use your result to express  $\dot{q} = \dot{q}(q, p)$ .
- b) Eliminate  $\dot{q}$  and write down the Hamiltonian  $\mathcal{H} = \mathcal{H}(q, p) = p\dot{q}(q, p) - \mathcal{L}(q, \dot{q}(q, p))$ .

Answer the parts below entirely by using the formal relation  $\mathcal{H}(q, p) = p\dot{q}(q, p) - \mathcal{L}(q, \dot{q}(q, p))$ .

- c) Find  $\partial\mathcal{H}/\partial p$  and show that the coefficient of  $\partial\dot{q}/\partial p$  is exactly zero, so that the final result is just  $\dot{q}$ .
- d) Find  $\partial\mathcal{H}/\partial q$  and show that the coefficient of  $\partial\dot{q}/\partial q$  is exactly zero. Then by using the EL equations for  $q$ , show that the final result is just  $-\dot{p}$ .

3. Consider the Atwood machine shown, where the pulley is a uniform disc of mass  $M$  and radius  $R$ .

- Using only  $x$  as your generalized coordinate, write down the Lagrangian, the generalized momentum  $p$ , and the Hamiltonian  $\mathcal{H} = p\dot{x} - \mathcal{L}$ .
- Find Hamilton's equations of motion.
- Use them to find the acceleration  $\ddot{x}$ .



4. In problem 1.3 we considered two particles, of mass  $m_1$  and  $m_2$  respectively, confined to the  $x$ -axis and connected by a spring with potential energy  $U = \frac{1}{2}kx^2$ , where  $x = x_1 - x_2 - l$  is the extension of the spring and  $l$  is its un-stretched length. There we found the Lagrangian for the reduced mass motion was:

$$\mathcal{L} = \frac{1}{2}\mu\dot{x}^2 - \frac{1}{2}kx^2.$$

- Let  $q$ , the generalized coordinate, be  $x$ . Find its canonical momentum  $p$ , and solve for  $\dot{q}$  as a function of  $q$  and  $p$ .
- Write down the Hamiltonian  $\mathcal{H}$  for the reduced system in terms of  $q$  and  $p$  alone.
- Write down Hamiltonian's equations of motion for the reduced system.
- These can be combined in the form  $\dot{\eta} = -i\omega\eta$ , where  $\eta = x + i\gamma p$  for some  $\gamma$  and  $\omega$  both positive. Find  $\gamma$  and  $\omega$ .
- What is the solution for  $x$  and  $p$ . Show that this is consistent with your solution for  $\dot{q}$  in a).

5. For two masses of (total mass  $M$ ) moving under the influence of gravity,  $\mu$ , the reduced mass of the system, is governed by the Lagrangian:

$$\mathcal{L} = \frac{1}{2}\mu \left( \dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2 \theta \dot{\phi}^2 \right) + \frac{G\mu M}{r}.$$

- Find the canonical momentum for each of the three coordinates  $r$ ,  $\theta$  and  $\phi$ .
- Are any of these conserved and, if so, which?
- Find the Hamiltonian.
- Write down Hamilton's equations of motion for all the variables involved.