PHY 4324 October 7, 2013

<u>Exam #1</u>

κτῆμά τε ἐς αἰεὶ μᾶλλον ἢ ἀγώνισμα
ἐς το παραχρῆμα ἀκούειν ξύγκειται

Thucydides I.22

(1) Suppose $\vec{J}(\vec{r})$ is constant in time but $\rho(\vec{r},t)$ is *not* — conditions that might prevail, for instance, during the charging of a capacitor.

a) Show that the charge density at any particular point is a linear function of time:

$$\rho(\vec{r},t) = \rho(\vec{r},0) + \dot{\rho}(\vec{r},0)t ,$$

where $\dot{\rho}(\vec{r},0)$ is the time derivative of ρ at $t = 0$. **(16 points)**

b) Suppose that the magnetic field is given at any time by the Biot-Savart law:

$$\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi} \int d^3r' \, \frac{\vec{J}(\vec{r}\,') \times (\vec{r} - \vec{r}\,')}{\|\vec{r} - \vec{r}\,'\|^3} .$$

Check that this obeys the Maxwell equation $\vec{\nabla} \cdot \vec{B} = 0$. **(16 points)**

c) Use the magnetic field given above, along with Faraday's law, to show that the electric field is the gradient of a scalar potential. **(16 points)**

d) Find an integral expression for the electric field like the one given above for the magnetic field. **(16 points)**

e) Show that fields obey the Ampere/Maxwell law. **(16 points)**

(2) Picture the electron as a uniformly charged spherical shell, with charge e and radius R, spinning at angular velocity $\omega \widehat{z}$. Recall that the electric and magnetic fields are:

$$\vec{E} = \frac{e \, \widehat{r} \theta(r - R)}{4\pi\epsilon_0 r^2} \qquad , \qquad \vec{B} = \frac{2}{3} \mu_0 \sigma \omega R \widehat{z} \, \theta(R - r) + \mu_0 \sigma \omega \frac{R^4}{3r^3} \Big(2\cos\theta \, \widehat{r} + \sin\theta \, \widehat{\theta} \Big) \theta(r - R) .$$

a) What is the total energy U contained in electromagnetic fields? **(10 points)**

b) What is the total angular momentum $\vec{L}$ contained in the fields? **(10 points)**

c) Suppose we set the energy and the angular momentum to their measured values:

$$U = m_e c^2 \qquad , \qquad \vec{L} = \frac{1}{2} \hbar \widehat{z} .$$

Express R and ω in terms of the fine structure constant $\alpha \equiv e^2/(4\pi\epsilon_0 \hbar c)$ and the Compton wavelength of the electron $\lambda_C \equiv \hbar/(m_e c)$. **(10 points)**

(3) Suppose

$$\vec{E}(r,\theta,\phi,t) = A \sin\theta \Big[\frac{\cos(kr - \omega t)}{r} - \frac{\sin(kr - \omega t)}{kr^2} \Big] \widehat{\phi} \qquad , \qquad \text{with } \frac{\omega}{k} = c .$$

a) What is the associated magnetic field $\vec{B}(r,\theta,\phi,t)$? **(20 points)**

b) Show that these fields obey Maxwell's equations in vacuum. **(20 points)**

c) Calculate the Poynting vector $\vec{S}(r,\theta,\phi,t)$. **(20 points)**

d) Integrate $\vec{S} \cdot d\vec{a}$ over a spherical surface. **(15 points)**