9/2/13
R.P. Woodard

Solutions to Exam #1

① * this was taken from problem 7.55 on page 339

(1a) * just use current conservation — $\dot{\rho} + \vec{\nabla}\cdot\vec{J} = 0$ — & the fact that $\vec{J}$ is constant $\Rightarrow \dot{\rho}(\vec{r},t) = -\vec{\nabla}\cdot\vec{J}(\vec{r}) = \text{constant} = \dot{\rho}(\vec{r},0)$

* now integrate $\int_0^t dt' \dot{\rho}(\vec{r},t') = \rho(\vec{r},t) - \rho(\vec{r},0) = -\vec{\nabla}\cdot\vec{J}(\vec{r})\, t$

$\therefore$ $\rho(\vec{r},t) = \rho(\vec{r},0) + \dot{\rho}(\vec{r},0)\,t = \rho(\vec{r},0) - \vec{\nabla}\cdot\vec{J}(\vec{r})\,t$ QED

(1b) * NB $\frac{\vec{r}-\vec{r}\,'}{\|\vec{r}-\vec{r}\,'\|^3} = -\vec{\nabla}\frac{1}{\|\vec{r}-\vec{r}\,'\|} \Rightarrow \vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\vec{\nabla}\times\int d^3r' \frac{\vec{J}(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|}$

* $\vec{\nabla}\cdot\vec{B} = 0$ because $\vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ for any $\vec{A}$

(1c) * $\dot{\vec{B}}(\vec{r}) = 0$

* $\vec{\nabla}\times\vec{E}(\vec{r},t) = -\dot{\vec{B}}(\vec{r}) = 0 \Rightarrow \vec{E}(\vec{r},t) = -\vec{\nabla}V(\vec{r},t)$

* cf Thm 1 on page 53

(1d) * $\vec{\nabla}\cdot\vec{E} = -\nabla^2 V = \rho/\epsilon_0 \Rightarrow V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r' \frac{\rho(\vec{r}\,',t)}{\|\vec{r}-\vec{r}\,'\|}$

* $\vec{E}(\vec{r},t) = -\frac{1}{4\pi\epsilon_0}\vec{\nabla}\int d^3r' \frac{\rho(\vec{r}\,',t)}{\|\vec{r}-\vec{r}\,'\|}$

(1e) * $\vec{\nabla}\times\vec{B}(\vec{r}) = \frac{\mu_0}{4\pi}\vec{\nabla}\times\left[\vec{\nabla}\times\int d^3r' \frac{\vec{J}(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|}\right] = \frac{\mu_0}{4\pi}\left\{\vec{\nabla}\,\vec{\nabla}\cdot\int d^3r' \frac{\vec{J}(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|} - \nabla^2\int d^3r' \frac{\vec{J}(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|}\right]$

* Recall $\nabla^2 \frac{1}{\|\vec{r}-\vec{r}\,'\|} = -4\pi\delta^3(\vec{r}-\vec{r}\,')$

* $\vec{\nabla}\cdot\int d^3r' \frac{\vec{J}(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|} = -\int d^3r'\, \vec{J}(\vec{r}\,')\cdot\vec{\nabla}'\frac{1}{\|\vec{r}-\vec{r}\,'\|} = +\int d^3r' \frac{\vec{\nabla}'\cdot\vec{J}(\vec{r}\,')}{\|\vec{r}-\vec{r}\,'\|} = -\int d^3r' \frac{\dot{\rho}(\vec{r}\,',0)}{\|\vec{r}-\vec{r}\,'\|}$

$\therefore$ $\vec{\nabla}\times\vec{B}(\vec{r}) = -\frac{\mu_0\vec{\nabla}}{4\pi}\int d^3r' \frac{\dot{\rho}(\vec{r}\,',0)}{\|\vec{r}-\vec{r}\,'\|} + \mu_0\vec{J}(\vec{r})$

* $-\frac{1}{c^2}\dot{\vec{E}}(\vec{r},t) = +\frac{\mu_0}{4\pi}\vec{\nabla}\int d^3r' \frac{\dot{\rho}(\vec{r}\,',t)}{\|\vec{r}-\vec{r}\,'\|} = \frac{\mu_0}{4\pi}\vec{\nabla}\int d^3r' \frac{\dot{\rho}(\vec{r}\,',0)}{\|\vec{r}-\vec{r}\,'\|}$

$\Rightarrow \vec{\nabla}\times\vec{B} - \frac{1}{c^2}\dot{\vec{E}} = \mu_0\vec{J}$

② * this was problem 8.11 on page 362 & was also HW #11

(2a) * $U_E \equiv \int d^3r \frac{1}{2}\epsilon_0\vec{E}\cdot\vec{E} = \frac{1}{2}\epsilon_0 * 4\pi\int_R^\infty dr\, r^2\left(\frac{e}{4\pi\epsilon_0 r^2}\right)^2 = \frac{e^2}{8\pi\epsilon_0 R}$

$\mu_0 = \frac{1}{\epsilon_0 c^2}$

* $U_B \equiv \int d^3r \frac{1}{2\mu_0}\vec{B}\cdot\vec{B} = \frac{1}{2\mu_0}\left(\frac{2}{3}\mu_0\sigma\omega R\right)^2\int d^3r\left\{4\Theta(R-r) + \frac{R^6}{r^6}\left[4\cos^2\theta + \sin^2\theta\right]\Theta(r-R)\right\}$

$= \frac{1}{18\epsilon_0 c^2}\left(\frac{e\omega}{4\pi R}\right)^2\left\{4*4\pi*\frac{1}{3}R^3 + 2\pi\int_R^\infty dr\, r^2\frac{R^6}{r^6}\int_0^\pi d\theta\sin\theta\left[1+3\cos^2\theta\right]\right\} = \frac{e^2}{8\pi\epsilon_0 R} * \frac{2}{9}\left(\frac{\omega R}{c}\right)^2$

$\sigma = \frac{e}{4\pi R^2}$

$\therefore$ $U = \frac{e^2}{8\pi\epsilon_0 R}\left[1 + \frac{2}{9}\left(\frac{\omega R}{c}\right)^2\right]$

9/12/13
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Solutions to Exam #1

$\hat r \times \hat\theta = \hat\phi$
$\hat\theta \times \hat\phi = \hat r$
$\hat\phi \times \hat r = \hat\theta$

(2b) $* \vec{\mathcal{P}}_{EM} \equiv \epsilon_0 \vec E \times \vec B = \frac{e^2}{16\pi^2\epsilon_0 R} \frac{\omega R}{c^2} * \frac{R^2}{3r^5} \sin\theta\, \hat\phi\, \Theta(r-R)$

$* \vec{\mathcal{L}}_{EM} \equiv \vec r \times \vec{\mathcal{P}}_{EM} = \frac{-e^2}{16\pi^2\epsilon_0 R} \frac{\omega R}{c^2} * \frac{R^2}{3r^4} \sin\theta\, \hat\theta\, \Theta(r-R)$

$*$ NB $\hat\theta = \cos\theta\cos\phi\,\hat x + \cos\theta\sin\phi\,\hat y - \sin\theta\,\hat z \xrightarrow{\int d\phi\, \hat\theta} -\sin\theta\,\hat z$

$* \vec L_{EM} \equiv \int d^3r\, \vec{\mathcal{L}}_{EM} = + \frac{e^2}{16\pi^2\epsilon_0 R} \frac{\omega R}{c^2} * 2\pi \int_R^\infty dr\, r^2 \frac{R^2}{3r^4} \int_0^\pi d\theta \sin^3\theta = \frac{e^2}{8\pi\epsilon_0 R} * \frac{4}{9} \frac{\omega R^2}{c^2} \hat z$

(2c) $* L_1 = \frac{1}{2}\hbar = \frac{e^2}{8\pi\epsilon_0 R} * \frac{4}{9} \frac{\omega R^2}{c^2} = \frac{2}{9} \alpha \frac{\omega R}{c} \hbar \Rightarrow \frac{\omega R}{c} = \frac{9}{4\alpha}$

$* U = m_e c^2 = \frac{\hbar c}{\lambda_c} = \frac{e^2}{8\pi\epsilon_0 R}\left[1 + \frac{2}{9}\left(\frac{\omega R}{c}\right)^2\right] = \alpha * \frac{\hbar c}{2R}\left[1 + \frac{2}{9}\left(\frac{9}{4\alpha}\right)^2\right]$

$\therefore \left\{ \begin{array}{l} \omega R = \frac{9c}{4\alpha} \\ R = \left(\frac{9}{16\alpha} + \frac{\alpha}{2}\right)\lambda_c \end{array} \right\} \Rightarrow \left\{ \begin{array}{l} R = \left(\frac{9}{16\alpha} + \frac{\alpha}{2}\right)\lambda_c \\ \omega = \frac{4c/\lambda_c}{1 + \frac{8}{9}\alpha^2} \end{array} \right\}$

(3) $*$ this was taken from problem 9.32, which was worked in class on Oct 4

$*$ NB it helps to note $\vec E = \hat z \times \vec\nabla \left[\frac{A \sin(kr - \omega t)}{kr}\right]$

(3a) $*$ get $\vec B$ from Faraday's Law: $\dot{\vec B} = -\vec\nabla \times \vec E \Rightarrow \vec B = \vec\nabla \times (\vec\nabla \times \hat z)\left[\frac{A}{\omega} \frac{\cos(kr - \omega t)}{kr}\right]$

$* \vec\nabla \times (\vec\nabla \times \hat z) = (\hat z \cdot \vec\nabla)\vec\nabla - \hat z \nabla^2 \xrightarrow{\text{on } f(r)} \left[\cos\theta \frac{\partial}{\partial r} - \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\right] \hat r \frac{\partial}{\partial r} - \hat z \nabla^2 = \frac{\hat r \cos\theta}{r} * -2\frac{\partial}{\partial r} + \frac{\hat\theta \sin\theta}{r}\left[\frac{\partial}{\partial r}\right]^2 r - \frac{\partial}{\partial r}$

$\Rightarrow \vec B = \hat r \cos\theta \frac{A}{c}\left[\frac{2\sin(kr-\omega t)}{kr^2} + \frac{2\cos(kr-\omega t)}{k^2r^3}\right] + \hat\theta \sin\theta \frac{A}{c}\left[-\frac{\cos(kr-\omega t)}{r} + \frac{\sin(kr-\omega t)}{kr^2} + \frac{\cos(kr-\omega t)}{k^2r^3}\right]$

(3b) $* \vec\nabla \cdot \vec E = 0$ is automatic because $\vec E = \vec\nabla \times$ something ✓

$* \vec\nabla \cdot \vec B = 0$ is automatic because $\vec B = \vec\nabla \times$ something ✓

$* \vec\nabla \times \vec B = -\vec\nabla \times \hat z\, \nabla^2 \left[\frac{A}{\omega} \frac{\cos(kr-\omega t)}{kr}\right] = + \frac{k^2}{\omega} \vec\nabla \times \hat z \left[\frac{A\cos(kr-\omega t)}{kr}\right]$

$* \frac{1}{c^2}\dot{\vec E} = -\frac{\omega}{c^2} \hat z \times \vec\nabla \left[\frac{A\cos(kr-\omega t)}{kr}\right] = + \frac{k^2}{\omega} \vec\nabla \times \hat z \left[\frac{A\cos(kr-\omega t)}{kr}\right] = \vec\nabla \times \vec B$ ✓

(3c) $* \vec S = \frac{1}{\mu_0} \vec E \times \vec B = \frac{-E_\phi B_\theta}{\mu_0} \hat r + \frac{E_\phi B_r}{\mu_0} \hat\theta \xrightarrow[\text{average}]{\text{time}} + \frac{A^2 \sin^2\theta}{2\mu_0 c r^2} \hat r$

(3d) $* d\vec a = r^2 d\theta \sin\theta\, d\phi\, \hat r$

$* d\vec a \cdot \vec S = \frac{A^2}{\mu_0 c} \sin^3\theta\, d\theta\, d\phi \left[\cos^2(u) - \frac{2}{kr}\sin(u)\cos(u) + \frac{1}{k^2r^2}\left(\sin^2(u) - \cos^2(u)\right) + \frac{\sin(u)\cos(u)}{k^3r^3}\right]$

$* \int d\vec a \cdot \vec S = \frac{4\pi}{3} \frac{A^2}{\mu_0 c} * \left[2\cos^2(u) - \frac{\sin(2u)}{kr} - \frac{2\cos(2u)}{k^2r^2} + \frac{\sin(2u)}{2k^3r^3}\right]$ $\quad u \equiv kr - \omega t$

$*$ NB the square-bracketed term time averages to 1