11/11/13
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Solutions to Exam #2

(1) * This was taken from problem 10.21

(1a) * just shift ϕ to $\phi - \omega t$ $\Rightarrow$ $\boxed{\rho(s,\phi,z,t) = \lambda_0 \left| \sin\left[\tfrac{1}{2}(\phi - \omega t)\right]\right| \delta(s-a)\,\delta(z)}$

* $\vec{J} = \vec{v}\rho \Rightarrow \boxed{\vec{J}(s,\phi,z,t) = \lambda_0 \omega a\, \hat{\phi}(\phi) \left|\sin\left[\tfrac{1}{2}(\phi-\omega t)\right]\right| \delta(s-a)\,\delta(z)}$

* NB $\hat{\phi}(\phi) \equiv -\sin(\phi)\,\hat{x} + \cos(\phi)\,\hat{y}$

(1b) * $t_{ret} \equiv t - \frac{1}{c}\|\vec{r} - \vec{r}\,'\| \longrightarrow \boxed{t - a/c}$

(1c) * $V(\vec{r},t) = \frac{1}{4\pi\epsilon_0}\int d^3r'\, \frac{\rho(\vec{r}\,', t_{ret})}{\|\vec{r}-\vec{r}\,'\|} \xrightarrow{\vec{r}=0} \frac{\lambda_0}{4\pi\epsilon_0}\int_0^{2\pi} d\phi \left|\sin\left[\tfrac{1}{2}(\phi - \omega t)\right]\right| = \boxed{\frac{\lambda_0}{\pi\epsilon_0}}$

(1d) * $\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\int d^3r'\, \frac{\vec{J}(\vec{r}\,', t_{ret})}{\|\vec{r}-\vec{r}\,'\|} \xrightarrow{\vec{r}=0} \frac{\mu_0 \lambda_0 \omega a}{4\pi}\int_0^{2\pi} d\phi \left|\sin\left[\tfrac{1}{2}(\phi - \omega t_{ret})\right]\right| \left[-\sin(\phi)\,\hat{x} + \cos(\phi)\,\hat{y}\right]$

* change variables to $\phi' = \phi - \omega t$ & expand $\sin(\phi' + \omega t)$ & $\cos(\phi' + \omega t)$

$$\Rightarrow \int_0^{2\pi} d\phi' \sin(\tfrac{1}{2}\phi')\left[-\sin(\phi' + \omega t_{ret})\,\hat{x} + \cos(\phi' + \omega t_{ret})\,\hat{y}\right] = \int_0^{2\pi} d\phi' \sin\left(\tfrac{\phi'}{2}\right)\left\{\begin{array}{l} -\left[\sin\left(\frac{\phi'}{2}\right)\cos(\omega t_{ret}) + \cos\left(\frac{\phi'}{2}\right)\sin(\omega t_{ret})\right]\hat{x} \\ + \left[\cos\left(\frac{\phi'}{2}\right)\cos(\omega t_{ret}) - \sin\left(\frac{\phi'}{2}\right)\sin(\omega t_{ret})\right]\hat{y} \end{array}\right\}$$

* $\sin(\phi') = 2\sin\left(\frac{\phi'}{2}\right)\cos\left(\frac{\phi'}{2}\right) \Rightarrow \int_0^{2\pi} d\phi' \sin\left(\frac{\phi'}{2}\right)\sin(\phi') = \frac{4}{3}\sin^3\left(\frac{\phi'}{2}\right)\Big|_0^{2\pi} = 0$

* $\cos(\phi') = 2\cos^2\left(\frac{\phi'}{2}\right) - 1 \Rightarrow \int_0^{2\pi} d\phi' \sin\left(\frac{\phi'}{2}\right)\cos(\phi') = \left[-\frac{4}{3}\cos^3\left(\frac{\phi'}{2}\right) + 2\cos\left(\frac{\phi'}{2}\right)\right]\Big|_0^{2\pi} = \frac{4}{3}$

$\therefore \boxed{\vec{A}(\vec{0},t) = \frac{\mu_0 \lambda_0 \omega a}{3\pi}\left[\sin(\omega t_{ret})\,\hat{x} - \cos(\omega t_{ret})\,\hat{y}\right]}$ where $t_{ret} = t - a/c$

(2) * This was taken from problem 11.21

* Take the origin as the equilibrium position
$\Rightarrow$ charge's world line is $\vec{w}(t) = -d\cos(\omega t)\,\hat{z}$ where $\omega = \sqrt{\frac{k}{m}}$

(2a) * Neglecting relativistic effects (!) the far fields are

$$\left.\begin{array}{l} \vec{E}(\vec{r},t) \longrightarrow \frac{\mu_0 q}{4\pi r}\,\hat{r} \times \left[\hat{r} \times \ddot{\vec{w}}\left(t - \frac{r}{c}\right)\right] \\ \vec{B}(\vec{r},t) \longrightarrow -\frac{\mu_0 q}{4\pi r c}\,\hat{r} \times \ddot{\vec{w}}\left(t - \frac{r}{c}\right) \end{array}\right\} \Rightarrow \vec{S}(\vec{r},t) \longrightarrow \frac{\mu_0 q^2}{16\pi^2 r^2 c}\|\hat{r} \times \ddot{\vec{w}}\|^2\,\hat{r}$$

* at R from the central axis $\vec{r} = -h\hat{z} + R\hat{s} \Rightarrow r^2 = h^2 + R^2$ & $\hat{r} = \frac{(-h\hat{z} + R\hat{s})}{\sqrt{h^2+R^2}}$

* $\hat{r} \times \ddot{\vec{w}} = -\frac{R d \omega^2 \cos(\omega t_{ret})}{\sqrt{h^2+R^2}}\,\hat{\phi} \Rightarrow \vec{S} \longrightarrow \frac{\mu_0 q^2}{16\pi^2 c}\,\frac{R^2 d^2 \omega^4 \cos^2(\omega t_{ret})}{(h^2+R^2)^2}\left(\frac{-h\hat{z} + R\hat{s}}{\sqrt{h^2+R^2}}\right)$

$$\left.\begin{array}{l} \text{* the floor has } d\vec{a} = -da\,\hat{z} \\ \text{* time averaging } \cos^2(\omega t_{ret}) \text{ gives } \frac{1}{2} \end{array}\right\} \Rightarrow \boxed{I \equiv \langle -\hat{z}\cdot\vec{S}\rangle = \frac{\mu_0 q^2 R^2 d^2 \omega^4 h}{32\pi^2 c\,[h^2+R^2]^{5/2}}}$$

(2b) * just multiply by $2\pi R\, dR$ & integrate

$$\Rightarrow P = \frac{\mu_0 q^2 d^2 \omega^4 h}{16\pi c}\int_0^\infty dR\,\frac{R^3}{[h^2+R^2]^{5/2}} \xrightarrow{R = h\tan\alpha} \frac{\mu_0 q^2 d^2 \omega^4}{16\pi c}\int_0^{\pi/2} d\alpha\,\sin^3(\alpha) = \boxed{\frac{\mu_0 q^2 d^2 \omega^4}{24\pi c}}$$

(2c)
$$\left.\begin{array}{l} \text{* same power radiated above } \Rightarrow \frac{dE}{dt} = -\frac{\mu_0 q^2 d^2 \omega^4}{12\pi c} \\ \text{* } E = \frac{1}{2}m\omega^2 d^2 = \frac{1}{2}k d^2 \Rightarrow \frac{dE}{dt} = k d\dot{d} \end{array}\right\} \Rightarrow \frac{\dot{d}}{d} = -\frac{\mu_0 q^2 \omega^4}{12\pi k c} \Rightarrow \boxed{\tau = \frac{12\pi k c}{\mu_0 q^2 \omega^4}}$$

Solutions to Exam #2

(3) * This was taken from problem 12.67

(3a) * there is no x-component force or initial velocity $\rightarrow x(t)=0$

* motion is in the y & z components

$\omega \equiv qB_0/m$

Factors of $\frac{1}{\sqrt{1-u^2/c^2}}$ prevent exact solution in terms of elementary functions

$$\left\{\begin{array}{l} \frac{d}{dt}\left(\frac{m u_y}{\sqrt{1-u^2/c^2}}\right) = qB_0 u_z \\ \frac{d}{dt}\left(\frac{m u_z}{\sqrt{1-u^2/c^2}}\right) = qE_0 - qB_0 v_y \end{array}\right\} \xrightarrow[\text{relativistic}]{\text{non-}} \left\{\begin{array}{l} \dot{u}_y = \omega u_z \\ \dot{u}_z = \frac{qE_0}{m} - \omega u_y \end{array}\right\} \rightarrow \left\{\begin{array}{l} \ddot{u}_y = -\omega^2 u_y + \frac{\omega q E_0}{m} \\ \ddot{u}_z = -\omega^2 u_z \end{array}\right\}$$

* the general solution for $u_y(t)$ & $u_z(t)$ is

$$\left\{\begin{array}{l} u_y(t) = \frac{E_0}{B_0} + \left[u_{y0} - \frac{E_0}{B_0}\right]\cos(\omega t) + u_{z0}\sin(\omega t) \\ u_z(t) = u_{z0}\cos(\omega t) + \left[\frac{E_0}{B_0} - u_{y0}\right]\sin(\omega t) \end{array}\right\} \xrightarrow[u_{y0}=0=u_{z0}]{} \left\{\begin{array}{l} u_y(t) = \frac{E_0}{B_0}\left[1-\cos(\omega t)\right] \\ u_z(t) = \frac{E_0}{B_0}\sin(\omega t) \end{array}\right\}$$

* integrate (with $y_0=0=z_0$) to get $y(t)$ & $z(t)$

$$\rightarrow \boxed{y(t) = \frac{E_0}{B_0}\left[t - \frac{\sin(\omega t)}{\omega}\right] \quad \& \quad z(t) = \frac{E_0}{B_0}\left[\frac{1-\cos(\omega t)}{\omega}\right]}$$

(3b) * For general $\vec{v}$ the S' frame fields are

$$\vec{E}' = \hat{\beta}\hat{\beta}\cdot\vec{E} + \gamma\left[\vec{E} - \hat{\beta}\hat{\beta}\cdot\vec{E} + \vec{v}\times\vec{B}\right] \xrightarrow[\vec{v}=v\hat{y}]{} 0 + \gamma\left[E_0\hat{z} - vB_0\hat{z}\right] \rightarrow \boxed{\vec{v} = \frac{E_0}{B_0}\hat{y}}$$

$$\vec{B}' = \hat{\beta}\hat{\beta}\cdot\vec{B} + \gamma\left[\vec{B} - \hat{\beta}\hat{\beta}\cdot\vec{B} - \frac{\vec{v}\times\vec{E}}{c^2}\right] \longrightarrow 0 + \gamma\left[B_0\hat{x} - \frac{vE_0}{c^2}\hat{x}\right] \rightarrow \boxed{\vec{B}' = \sqrt{B_0^2 - \frac{E_0^2}{c^2}}\,\hat{x}}$$

* NB we can get B' from the fact that $B^2 - E^2/c^2$ is invariant

(3c) * NB $\vec{u}_0 = -\vec{v} = -\frac{E_0}{B_0}\hat{y} \rightarrow \Gamma' = \frac{1}{\sqrt{1-u^2/c^2}} = \frac{1}{\sqrt{1-E_0^2/c^2B_0^2}}$ is constant

$\Gamma' = \gamma$

* the S' eqns ~~are~~ for u_y' & u_z' are — where $\omega' \equiv \frac{qB'}{m\Gamma'} = \frac{qB'}{m\gamma}$

$$\left\{\begin{array}{l} \frac{d}{dt'}(m\Gamma' u_y') = qB' u_z' \\ \frac{d}{dt'}(m\Gamma' u_z') = -qB' u_y' \end{array}\right\} \rightarrow \left\{\begin{array}{l} \dot{u}_y' = \omega' u_z' \\ \dot{u}_z' = -\omega' u_y' \end{array}\right\} \rightarrow \left\{\begin{array}{l} \ddot{u}_y' = -\omega'^2 u_y' \\ \ddot{u}_z' = -\omega'^2 u_z' \end{array}\right\}$$

* the general solution for $u_y'(t')$ & $u_z'(t')$ is

$$\left\{\begin{array}{l} u_y'(t') = u_{y0}'\cos(\omega' t') + u_{z0}'\sin(\omega' t') \\ u_z'(t') = u_{z0}'\cos(\omega' t') - u_{y0}'\sin(\omega' t') \end{array}\right\} \xrightarrow[u_{z0}'=0]{u_{y0}' = -\frac{E_0}{B_0}} \left\{\begin{array}{l} u_y'(t') = -\frac{E_0}{B_0}\cos(\omega' t') \\ u_z'(t') = \frac{E_0}{B_0}\sin(\omega' t') \end{array}\right\}$$

* integrate (with $y_0'=0=z_0'$) to get $y'(t')$ & $z'(t')$

$$\rightarrow \boxed{y'(t') = -\frac{E_0}{B_0}\frac{\sin(\omega' t')}{\omega'} \quad \& \quad z'(t') = \frac{E_0}{B_0}\left[\frac{1-\cos(\omega' t')}{\omega'}\right]} \text{ of course } \boxed{x'(t')=0}$$

(3d) * just Lorentz transform back to S

the eqn which determines $t'(t)$ is

$$\left\{\begin{array}{l} x(t) = 0 \\ y = \gamma(y' + vt') \\ z = z' \end{array}\right\} \rightarrow \boxed{\begin{array}{l} x = 0 \\ y = \frac{\gamma E_0}{B_0}\left[t' - \frac{\sin(\omega' t')}{\omega'}\right] \\ z = \frac{E_0}{B_0}\left[\frac{1-\cos(\omega' t')}{\omega'}\right] \end{array}} \qquad \boxed{t = \gamma\left[t' - \frac{v^2}{c^2}\frac{\sin(\omega' t')}{\omega'}\right]}$$