8/27/13
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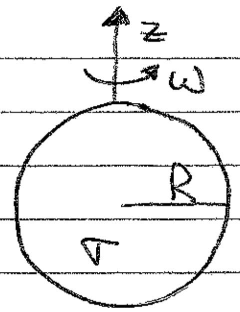
Solution to Problem 8.3



$\vec{K} = R\omega\sigma \sin\theta\,\hat{\phi}$

$\boxed{\text{Recall Problem 5.42}}$ $\quad \vec{B} = \vec{\nabla}\psi$

* use a scalar potential inside & out

* $0 < r < R \rightarrow \psi_{in} = a z = a r\cos\theta$

$\rightarrow B_{inside} = a\hat{z} = a[\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}]$

* $R < r < \infty \rightarrow \psi_{out} = \frac{b\cos\theta}{r^2}$

$\rightarrow \vec{B}_{outside} = \frac{-b}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right]$

$$\left.\begin{array}{l} * \ \hat{r}\cdot\Delta\vec{B} = 0 \rightarrow -a - \frac{2b}{R^2} = 0 \\ * \ \hat{r}\times\Delta\vec{B} = \mu_0\vec{K} \rightarrow +a - \frac{b}{R^3} = \mu_0\sigma\omega R \end{array}\right\} \rightarrow \left\{\begin{array}{l} a = \frac{2}{3}\mu_0\sigma\omega R \\ b = -\frac{1}{3}\mu_0\sigma\omega R^4 \end{array}\right\}$$

$$\therefore \left\{\begin{array}{l} \vec{B}_{inside} = \frac{2}{3}\mu_0\sigma\omega R\,\hat{z} \\ \vec{B}_{outside} = \frac{1}{3}\mu_0\sigma\omega\frac{R^4}{r^3}\left[2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right] \end{array}\right\}$$

$\boxed{\text{Use Magnetic Part of } T^{ij}}$ $\quad \hat{z}^i\hat{z}^j$

* $T^{ij}_{inside} = \frac{1}{9}\mu_0\sigma^2\omega^2R^2\left[4\delta^i_3\delta^j_3 - \frac{4}{2}\delta^{ij}\right]$

* $T^{ij}_{outside} = \frac{1}{9}\mu_0\sigma^2\omega^2R^2\left[(2\cos\theta\,\hat{r}^i + \sin\theta\,\hat{\theta}^i)(2\cos\theta\,\hat{r}^j + \sin\theta\,\hat{\theta}^j) - \frac{1}{2}\delta^{ij}(4\cos^2\theta + \sin^2\theta)\right]$

$\boxed{\text{The area element}}$

* $d\vec{a}_{inside} = -R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$

* $d\vec{a}_{outside} = +R^2\sin\theta\,d\theta\,d\phi\,\hat{r}$

$\boxed{T^{ij}da_j}$ $\quad \hat{z} = \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}$

* $(T^{ij}da_j)_{inside} = -\frac{1}{9}\mu_0\sigma^2\omega^2R^4\sin\theta\,d\theta\,d\phi\left[4\hat{z}^i\cos\theta - 2\hat{r}^i\right]$

* $(T^{ij}da_j)_{outside} = \frac{1}{9}\mu_0\sigma^2\omega^2R^4\sin\theta\,d\theta\,d\phi\left[2\cos\theta(2\cos\theta\,\hat{r}^i + \sin\theta\,\hat{\theta}^i) - \frac{1}{2}\hat{r}^i(4\cos^2\theta + \sin^2\theta)\right]$

$$\therefore \vec{F} = \frac{1}{9}\mu_0\sigma^2\omega^2R^4\int_0^{2\pi}d\phi\int_0^{\pi/2}d\theta\sin\theta\left[\frac{3}{2}\sin\theta\,\hat{r} + 6\sin\theta\cos\theta\,\hat{\theta}\right]$$

$$\left.\begin{array}{l} * \text{ Recall } \hat{r} = \sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z} \\ \qquad\quad \hat{\theta} = \cos\theta\cos\phi\,\hat{x} + \cos\theta\sin\phi\,\hat{y} - \sin\theta\,\hat{z} \end{array}\right\} \rightarrow \int_0^{2\pi}d\phi \text{ kills the } \hat{x}\ \&\ \hat{y} \text{ terms}$$

$$\rightarrow \vec{F} = \frac{1}{9}\mu_0\sigma^2\omega^2R^4 \times -\frac{9}{4}\pi\,\hat{z} = -\frac{\pi}{4}\mu_0\sigma^2\omega^2R^4\,\hat{z}$$