9/17/13
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Solution to Problem 8.11

$\boxed{\text{The Fields}}$

$$* \sigma = \frac{e}{4\pi R^2} \Rightarrow \vec{E} = \begin{cases} 0 \le r \le R \Rightarrow 0 \\ R \le r < \infty \Rightarrow \frac{e\hat{r}}{4\pi\epsilon_0}\frac{1}{r^2} \end{cases} \;\&\; \vec{B} = \begin{cases} 0 \le r \le R \Rightarrow \frac{e}{4\pi\epsilon_0}\frac{\omega R}{c^2} * \frac{2}{3R^2}\hat{z} \\ R \le r < \infty \Rightarrow \frac{e}{4\pi\epsilon_0}\frac{\omega R}{c}\left[\frac{2}{3}\frac{R}{r^3}\cos\theta\,\hat{r} + \frac{R}{3r^3}\sin\theta\,\hat{\theta}\right] \end{cases}$$

(a) $* u_E = \frac{\epsilon_0}{2}E^2 \;\&\; u_B = \frac{1}{2\mu_0}B^2 = \frac{\epsilon_0}{2}c^2B^2$

$* \text{Inside} \Rightarrow U_E = 0 \quad \& \; U_B = \frac{e^2}{8\pi\epsilon_0 R} * \left(\frac{\omega R}{c}\right)^2 * \frac{4}{27}$

$* \text{Outside} \Rightarrow U_E = \frac{e^2}{8\pi\epsilon_0 R} \;\&\; U_B = \frac{e^2}{8\pi\epsilon_0 R} * \left(\frac{\omega R}{c}\right)^2 * \frac{2}{27}$

$\Rightarrow \boxed{U_{total} = \frac{e^2}{8\pi\epsilon_0 R}\left[1 + \frac{2}{9}\left(\frac{\omega R}{c}\right)^2\right]}$

(b) $* \text{first compute } \vec{\wp}_{em} = \epsilon_0 \vec{E} \times \vec{B} = \begin{cases} 0 \le r \le R \Rightarrow 0 \\ R \le r < \infty \Rightarrow \frac{e^2}{16\pi^2\epsilon_0}\frac{\omega R}{c^2} * \frac{R}{3r^5}\sin\theta\,\hat{\phi} \end{cases}$

* NB this integrates to zero!

$* \vec{\ell}_{em} \equiv \vec{r} \times \vec{\wp}_{em} = \frac{e^2}{16\pi^2\epsilon_0}\frac{\omega R}{c^2} * \Theta(r-R)\frac{R}{3r^4}\sin\theta * -\hat{\theta}$

$* \text{NB} \int_0^\pi d\theta \sin\theta \int_0^{2\pi} d\phi \left(-\sin\theta\,\hat{\theta}\right) = 2\pi\hat{z}\int_0^\pi d\theta \sin^3\theta = \frac{8\pi}{3}\hat{z}$

$* \int_R^\infty dr\, r^2 * \frac{R}{3r^4} = 1/3 \Rightarrow \boxed{\vec{L}_1 = \frac{e^2}{8\pi\epsilon_0 R} * \frac{\omega R^2}{c^2} * \frac{4}{9}\hat{z}}$

(c) $* U = \frac{e^2}{8\pi\epsilon_0 R}\left[1 + \frac{2}{9}\left(\frac{\omega R}{c}\right)^2\right] = m_e c^2$

$* L_1 = \frac{e^2}{8\pi\epsilon_0 R} * \frac{4\omega R^2}{9c^2} = \frac{\hbar}{2}$

Fine structure constant

compton wavelength of electron

$\boxed{\text{Re-Express in terms of } \alpha = \frac{e^2}{4\pi\epsilon_0\hbar c} \simeq \frac{1}{137} \;\&\; \lambda_c \equiv \frac{\hbar}{m_e c} \simeq 3.86 \times 10^{-13}\,\text{m}}$

$* \alpha\left[1 + \frac{2}{9}\left(\frac{\omega R}{c}\right)^2\right] = 2\frac{R}{\lambda_c} \nearrow \boxed{\frac{R}{\lambda_c} = \frac{9}{16\alpha} + \frac{1}{2}\alpha} \simeq 77$

$* \alpha \cdot \frac{4}{9}\frac{\omega R}{c} = 1 \Rightarrow \boxed{\frac{\omega R}{c} = \frac{9}{4\alpha}} \simeq 308$

$\therefore \boxed{\begin{array}{l} R \simeq 3.0 \times 10^{-11}\,\text{m} \\ \omega \simeq 3.1 \times 10^{21}\,\text{Hz} \end{array}} \Rightarrow \boxed{\omega R = \frac{9}{4\alpha}c \simeq 308c \simeq 9.3 \times 10^{10}\,\text{m/s}}$

* this is not a successful model – but it is intriguing!