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Solution to Problem 9.11

$* \; f(\vec{r},t) = A\cos(\vec{k}\cdot\vec{r} - \omega t + \delta_a) \Rightarrow \tilde{f}(\vec{r},t) = A e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta_a)}$

$* \; g(\vec{r},t) = B\cos(\vec{k}\cdot\vec{r} - \omega t + \delta_b) \Rightarrow \tilde{g}(\vec{r},t) = B e^{i(\vec{k}\cdot\vec{r} - \omega t + \delta_b)}$

$\therefore \; \tilde{f}(\vec{r},t)\,\tilde{g}^*(\vec{r},t) = AB e^{i(\delta_a - \delta_b)} \Rightarrow \frac{1}{2}\mathrm{Re}\left[\tilde{f}\,\tilde{g}^*\right] = \frac{1}{2}AB\cos(\delta_a - \delta_b)$

$*$ the time-averaged product is

$$\langle fg \rangle = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\, f(\vec{r},t)\, g(\vec{r},t) = AB \lim_{T\to\infty}\frac{1}{T}\int_0^T dt \cos(\vec{k}\cdot\vec{r} - \omega t + \delta_a)\cos(\vec{k}\cdot\vec{r} - \omega t + \delta_b)$$

$*$ NB $\theta \equiv \vec{k}\cdot\vec{r} - \omega t \Rightarrow \cos(\theta + \delta_a)\cos(\theta + \delta_b) = \frac{1}{2}\cos(2\theta + \delta_a + \delta_b) + \frac{1}{2}\cos(\delta_a - \delta_b)$

$$\therefore \langle fg \rangle = \lim_{T\to\infty}\frac{AB}{2T}\int_0^T dt \left\{\cos\left[2\vec{k}\cdot\vec{r} - 2\omega t + \delta_a + \delta_b\right] + \cos(\delta_a - \delta_b)\right\}$$

$$= \lim_{T\to\infty}\frac{AB}{2T}\left\{\frac{\sin\left[2\vec{k}\cdot\vec{r} - 2\omega t + \delta_a + \delta_b\right]}{-2\omega} + T\cos(\delta_a - \delta_b)\right\}$$

$$= \frac{1}{2}AB\cos(\delta_a - \delta_b) = \frac{1}{2}\mathrm{Re}\left[\tilde{f}\,\tilde{g}^*\right] \quad \text{QED}$$