9/25/13
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Solution to Problem 9.26

(a) * $\vec{E} = \left[E_x(x,y)\hat{x} + E_y(x,y)\hat{y} + E_z(x,y)\hat{z}\right] e^{i(kz-\omega t)} \Rightarrow \dot{\vec{E}} = -i\omega\vec{E}$

$$\Rightarrow \vec{\nabla}\times\vec{E} = \left[\hat{z}\frac{\partial E_y}{\partial x} - \hat{y}\frac{\partial E_z}{\partial x} - \hat{z}\frac{\partial E_x}{\partial y} + \hat{x}\frac{\partial E_z}{\partial y} + ikE_x\hat{y} - ikE_y\hat{x}\right] e^{i(kz-\omega t)}$$

$$= \left[\hat{x}\left(\frac{\partial E_z}{\partial y} - ikE_y\right) + \hat{y}\left(ikE_x - \frac{\partial E_z}{\partial x}\right) + \hat{z}\left(\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y}\right)\right] e^{i(kz-\omega t)}$$

* $\vec{B} = \left[B_x(x,y)\hat{x} + B_y(x,y)\hat{y} + B_z(x,y)\hat{z}\right] e^{i(kz-\omega t)} \Rightarrow \dot{\vec{B}} = -i\omega\vec{B}$

$$\Rightarrow \vec{\nabla}\times\vec{B} = \left[\hat{x}\left(\frac{\partial B_z}{\partial y} - ikB_y\right) + \hat{y}\left(ikB_x - \frac{\partial B_z}{\partial x}\right) + \hat{z}\left(\frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y}\right)\right] e^{i(kz-\omega t)}$$

$\boxed{\vec{\nabla}\times\vec{E} = -\dot{\vec{B}}}$ $\qquad\qquad$ $\boxed{\vec{\nabla}\times\vec{B} = \frac{1}{c^2}\dot{\vec{E}}}$

* $\hat{z}$ comp $\Rightarrow \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = i\omega B_z$ (i) $\qquad \frac{\partial B_y}{\partial x} - \frac{\partial B_x}{\partial y} = -\frac{i\omega}{c^2}E_z$ (iv)

* $\hat{x}$ comp $\Rightarrow \frac{\partial E_z}{\partial y} - ikE_y = i\omega B_x$ (ii) $\qquad \frac{\partial B_z}{\partial y} - ikB_y = -\frac{i\omega}{c^2}E_x$ (v)

* $\hat{y}$ comp $\Rightarrow ikE_x - \frac{\partial E_z}{\partial x} = i\omega B_y$ (iii) $\qquad ikB_x - \frac{\partial B_z}{\partial x} = -\frac{i\omega}{c^2}E_y$ (vi)

$\boxed{\text{Deriving 9.180}}$

$$\left.\begin{array}{l} \text{*(iii)} \Rightarrow ikE_x - i\omega B_y = \frac{\partial E_z}{\partial x} \\ \text{*(v)} \Rightarrow \frac{i\omega}{c^2}E_x - ikB_y = -\frac{\partial B_z}{\partial y} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} E_x = \frac{i\left[k\frac{\partial E_z}{\partial x} + \omega\frac{\partial B_z}{\partial y}\right]}{\omega^2/c^2 - k^2} \\ B_y = \frac{i\left[\frac{\omega}{c^2}\frac{\partial E_z}{\partial x} + k\frac{\partial B_z}{\partial y}\right]}{\omega^2/c^2 - k^2} \end{array}\right\}$$

$$\left.\begin{array}{l} \text{*(ii)} \Rightarrow ikE_y + i\omega B_x = \frac{\partial E_z}{\partial y} \\ \text{*(vi)} \Rightarrow \frac{i\omega}{c^2}E_y + ikB_x = \frac{\partial B_z}{\partial x} \end{array}\right\} \Rightarrow \left\{\begin{array}{l} E_y = i\left[-\omega\frac{\partial B_z}{\partial x} + k\frac{\partial E_z}{\partial y}\right]/(\omega^2/c^2 - k^2) \\ B_x = i\left[k\frac{\partial B_z}{\partial x} - \frac{\omega}{c^2}\frac{\partial E_z}{\partial y}\right]/(\omega^2/c^2 - k^2) \end{array}\right\}$$

(b)

$$\left.\begin{array}{l} * \vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + ikE_z = 0 \\ * \frac{\partial E_x}{\partial x} = i\left[\frac{\omega^2}{c^2} - k^2\right]^{-1}\left[k\frac{\partial^2 E_z}{\partial x^2} + \cancel{\omega\frac{\partial^2 B_z}{\partial x\partial y}}\right] \\ * \frac{\partial E_y}{\partial y} = i\left[\frac{\omega^2}{c^2} - k^2\right]^{-1}\left[\cancel{-\omega\frac{\partial^2 B_z}{\partial x\partial y}} + k\frac{\partial^2 E_z}{\partial y^2}\right] \end{array}\right\} \Rightarrow \frac{ik}{\frac{\omega^2}{c^2} - k^2}\left[\frac{\partial^2 E_z}{\partial x^2} + \frac{\partial^2 E_z}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k^2\right)E_z\right] = 0$$

$$\left.\begin{array}{l} * \vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} + ikB_z = 0 \\ * \frac{\partial B_x}{\partial x} = i\left[\frac{\omega^2}{c^2} - k^2\right]^{-1}\left[k\frac{\partial^2 B_z}{\partial x^2} - \frac{\omega}{c^2}\frac{\partial^2 E_z}{\partial x\partial y}\right] \\ * \frac{\partial B_y}{\partial y} = i\left[\frac{\omega^2}{c^2} - k^2\right]^{-1}\left[\frac{\omega}{c^2}\frac{\partial^2 E_z}{\partial x\partial y} + k\frac{\partial^2 B_z}{\partial y^2}\right] \end{array}\right\} \Rightarrow \frac{ik}{\frac{\omega^2}{c^2} - k^2}\left[\frac{\partial^2 B_z}{\partial x^2} + \frac{\partial^2 B_z}{\partial y^2} + \left(\frac{\omega^2}{c^2} - k^2\right)B_z\right] = 0$$

$$\text{* from (i)} \left\{\begin{array}{l} \frac{\partial E_y}{\partial x} = i\left[\frac{\omega^2}{c^2} - k^2\right]^{-1}\left[k\frac{\partial^2 E_z}{\partial x\partial y} - \omega\frac{\partial^2 B_z}{\partial x^2}\right] \\ -\frac{\partial E_x}{\partial y} = i\left[\frac{\omega^2}{c^2} - k^2\right]^{-1}\left[-k\frac{\partial^2 E_z}{\partial x\partial y} - \omega\frac{\partial^2 B_z}{\partial y^2}\right] \end{array}\right\} = \frac{-i\omega}{\left[\frac{\omega^2}{c^2} - k^2\right]}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]B_z = i\omega B_z$$

$$\text{* from (iv)} \left\{\begin{array}{l} \frac{\partial B_y}{\partial x} = i\left[\frac{\omega^2}{c^2} - k^2\right]^{-1}\left[k\frac{\partial^2 B_z}{\partial x\partial y} + \frac{\omega}{c^2}\frac{\partial^2 E_z}{\partial x^2}\right] \\ -\frac{\partial B_x}{\partial y} = i\left[\frac{\omega^2}{c^2} - k^2\right]^{-1}\left[-k\frac{\partial^2 B_z}{\partial x\partial y} + \frac{\omega}{c^2}\frac{\partial^2 E_z}{\partial y^2}\right] \end{array}\right\} = \frac{i\omega/c^2}{\left[\frac{\omega^2}{c^2} - k^2\right]}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right]E_z = -\frac{i\omega}{c^2}E_z$$