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Solution to Problem 9.32

$$\tilde{A} = A_R + i A_I$$

* From eqn (9.20) we have $\tilde{f}(z,t) = \int_{-\infty}^{\infty} dk\, e^{i(kz-\omega t)} \tilde{A}(k)$

* However, note that we can only use the real part

$$f(z,t) = \mathrm{Re}\left[\tilde{f}(z,t)\right] = \int_{-\infty}^{\infty} dk \left\{ \cos(kz-\omega t) A_R(k) - \sin(kz-\omega t) A_I(k) \right\}$$

* $f(z,0) = \int_{-\infty}^{\infty} dk \left\{ \cos(kz) A_R(k) - \sin(kz) A_I(k) \right\}$

* $\frac{i}{\omega} \dot{f}(z,0) = + \int_{-\infty}^{\infty} dk \left\{ i \sin(kz) A_R(k) + i \cos(kz) A_I(k) \right\}$

$\therefore$ $f(z,0) + \frac{i}{\omega} \dot{f}(z,0) = \int_{-\infty}^{\infty} dk \left\{ e^{ikz} A_R(k) + i e^{ikz} A_I(k) \right\} = \int_{-\infty}^{\infty} dk\, e^{ikz} \tilde{A}(k)$

* Now use the Fourier Inversion Thm

$$\Rightarrow \tilde{A}(k) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dz\, e^{-ikz} \left[f(z,0) + \frac{i}{\omega} \dot{f}(z,0) \right] \qquad \text{QED}$$

* NB it's simpler to derive this by first Fourier transforming the differential eqn $\frac{1}{v^2} \ddot{f} = f'' \Rightarrow \frac{1}{v^2} \ddot{F} = -k^2 F$

$$\Rightarrow F(k,t) = F(k,0) \cos(\underbrace{vk}_{\omega} t) + \frac{\dot{F}(k,0)}{vk} \sin(vkt)$$

$$\Rightarrow f(z,t) = \int_{-\infty}^{\infty} dk\, e^{ikz} \Big[F(k,0) \underbrace{\cos(\omega t)}_{\frac{1}{2}[e^{i\omega t} + e^{-i\omega t}]} + \frac{\dot{F}(k,0)}{\omega} \underbrace{\sin(\omega t)}_{\frac{i}{2}[-e^{i\omega t} + e^{-i\omega t}]} \Big]$$

$$= \int_{-\infty}^{\infty} dk \Big\{ e^{i(kz-\omega t)} \frac{1}{2} \Big[F(k,0) + \frac{i}{\omega} \dot{F}(k,0) \Big] + e^{i(kz+\omega t)} \frac{1}{2} \Big[F(k,0) - \frac{i}{\omega} \dot{F}(k,0)$$