10/11/13
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Solution to Problem 10.10

* First write out the vector potential at a general point $\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$

$$\vec{A}(\vec{r},t) = \frac{\mu_0}{4\pi}\Big\{\int_{-b}^{-a} dx' \frac{I(t_{ret})\hat{x}}{\sqrt{(x-x')^2+y^2+z^2}} + \int_{\pi}^{0} d\phi\, a \frac{I(t_{ret})[-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}]}{\sqrt{(x-a\cos\phi)^2+(y-a\sin\phi)^2+z^2}} + \int_a^b dx' \frac{I(t_{ret})\hat{x}}{\sqrt{(x-x')^2+y^2+z^2}}$$

* NB $t_{ret} \equiv t - \frac{1}{c}\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}$

$$+ \int_0^{\pi} d\phi\, b \frac{I(t_{ret})[-\sin\phi\,\hat{x} + \cos\phi\,\hat{y}]}{\sqrt{(x-b\cos\phi)^2+(y-b\sin\phi)^2+z^2}}\Big\}$$

* Now specialize to $\vec{r} = 0$

$$\Rightarrow \vec{A}(\vec{0},t) = \frac{\mu_0}{4\pi}\Big\{\int_{-b}^{-a} dx' \frac{I(t-\frac{|x'|}{c})}{|x'|}\hat{x} + \int_{\pi}^{0} d\phi\, \cancel{a} \frac{I(t-\frac{a}{c})[-\sin\phi\,\hat{x}+\cos\phi\,\hat{y}]}{\cancel{a}} + \int_a^b dx \frac{I(t-\frac{x}{c})}{x}\hat{x} + \int_0^{\pi} d\phi\, \cancel{b} \frac{I(t-\frac{b}{c})}{\cancel{b}}\begin{bmatrix}-\sin\phi\,\hat{x}\\ +\cos\phi\,\hat{y}\end{bmatrix}$$

$$= \frac{\mu_0}{4\pi}\Big\{2\int_a^b dx' \frac{I(t-\frac{x'}{c})}{x'}\hat{x} + 2\left[I(t-\tfrac{a}{c}) - I(t-\tfrac{b}{c})\right]\hat{x}\Big\}$$

$$= \frac{\mu_0}{2\pi}\Big\{\int_a^b \frac{dx}{x} I(t-\tfrac{x}{c}) + I(t-\tfrac{a}{c}) - I(t-\tfrac{b}{c})\Big\}\hat{x}$$

* Finally we use $I(t) = kt$

$$\Rightarrow \vec{A}(0,t) = \frac{\mu_0 k}{2\pi}\Big\{\int_a^b dx\Big(\frac{t - x/c}{x}\Big) + \cancel{t} - \frac{a}{c} - \cancel{t} + \frac{b}{c}\Big\}\hat{x}$$

$$= \frac{\mu_0 k}{2\pi}\Big\{t\ln\Big(\frac{b}{a}\Big) - \frac{b}{c} + \frac{a}{c} - \frac{a}{c} + \frac{b}{c}\Big\}\hat{x}$$

$$= \frac{\mu_0 k t}{2\pi}\ln(b/a)\hat{x} = \frac{\mu_0}{2\pi}\ln\Big(\frac{b}{a}\Big)\hat{x} * I(t)$$

* Because $\rho(\vec{r},t) = 0 \Rightarrow V(\vec{r},t) = 0$

* $\vec{E}(\vec{r},t) = -\vec{\nabla}V(\vec{r},t) - \dot{\vec{A}}(\vec{r},t) \Rightarrow \vec{E}(0,t) = -\frac{\mu_0 k}{2\pi}\ln\Big(\frac{b}{a}\Big)\hat{x}$

* Of course $\vec{E} \neq 0$ because $\dot{\vec{B}} \neq 0$

* We can't get $\vec{B}(0,t)$ because $\vec{B}(\vec{r},t) = \vec{\nabla}\times\vec{A}(\vec{r},t)$
& we don't know how $\vec{A}(\vec{r},t)$ depends upon $\vec{r}$