10/14/15
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Solution to Problem 10.26

$$\text{*Fields of } q_1 \rightarrow \left\{ \begin{aligned} \vec{E}_1(\vec{r},t) &= \frac{q_1}{4\pi\epsilon_0}\frac{\vec{r}}{r^3} \\ \vec{B}_1(\vec{r},t) &= 0 \end{aligned} \right\}$$

* to get the fields of q_2, recall my brilliant & error-free lecture of Oct. 14 with $\vec{w}(t) = \hat{z}vt$

$$\therefore \left\{ \begin{aligned} V_2(\vec{r},t) &= \frac{q_2}{4\pi\epsilon_0}\frac{1}{\sqrt{(1-\beta^2)(x^2+y^2)+(z-vt)^2}} \\ \vec{A}_2(\vec{r},t) &= \frac{q_2}{4\pi\epsilon_0 c}\frac{\beta\hat{z}}{\sqrt{(1-\beta^2)(x^2+y^2)+(z-vt)^2}} \end{aligned} \right\} \rightarrow \left\{ \begin{aligned} \vec{E}_2(\vec{r},t) &= \frac{q_2}{4\pi\epsilon_0}\frac{(1-\beta^2)[x\hat{x}+y\hat{y}+(z-vt)\hat{z}]}{[(1-\beta^2)(x^2+y^2)+(z-vt)^2]^{3/2}} \\ \vec{B}_2(\vec{r},t) &= \frac{q_2\beta}{4\pi\epsilon_0 c}\frac{(1-\beta^2)(x\hat{y}-y\hat{x})}{[(1-\beta^2)(x^2+y^2)+(z-vt)^2]^{3/2}} \end{aligned} \right\}$$

(a) $*\ \vec{F}_{12}(t) = q_2\left[\vec{E}_1(vt\hat{z},t) + \vec{v}\times\vec{B}_1(vt\hat{z},t)\right] = \frac{q_1 q_2}{4\pi\epsilon_0}\frac{\hat{z}}{v^2t^2}$

(b) $*\ \vec{F}_{21}(t) = q_1\left[\vec{E}_2(\vec{0},t) + \vec{0}\right] = -\frac{q_1 q_2}{4\pi\epsilon_0}\frac{(1-\beta^2)\hat{z}}{v^2t^2}$

$\vec{F}_{12} + \vec{F}_{21} = \frac{q_1 q_2 \hat{z}}{4\pi\epsilon_0 c^2 t^2} \neq 0$

$\therefore$ the 3rd Law is violated

(c) $*\ \vec{\wp}_{em} \equiv \epsilon_0(\vec{E}_1+\vec{E}_2)\times(\cancel{\vec{B}_1}+\vec{B}_2) = \epsilon_0\vec{E}_1\times\vec{B}_2 + \epsilon_0\vec{E}_2\times\vec{B}_2$

* ignore $\epsilon_0\vec{E}_2\times\vec{B}_2$ whose integral is constant

$$*\ \epsilon_0\vec{E}_1\times\vec{B}_2 = \frac{q_1q_2\beta(1-\beta^2)}{16\pi^2\epsilon_0 c}\frac{[-z(x\hat{x}+y\hat{y})+(x^2+y^2)\hat{z}]}{r^3[(1-\beta^2)(x^2+y^2)+(z-vt)^2]^{3/2}}$$

ϕ integral gives zero

$$*\ \epsilon_0\int d^3r\,\vec{E}_1\times\vec{B}_2 = \frac{q_1q_2\beta(1-\beta^2)}{16\pi^2\epsilon_0 c}\int_0^\infty dr\,r^2\int_0^\pi d\theta\sin\theta\int_0^{2\pi}d\phi\,\frac{[\cancel{-z(x\hat{x}+y\hat{y})}+r^2\sin^2\theta\,\hat{z}]}{r^3[(1-\beta^2)r^2\sin^2\theta+(r\cos\theta-vt)^2]^{3/2}}$$

$$= \frac{q_1q_2\beta(1-\beta^2)}{8\pi\epsilon_0 c}\hat{z}\int_0^\pi d\theta\sin^3\theta\int_0^\infty dr\,\frac{r}{[(1-\beta^2\sin^2\theta)r^2-2vt\,r\cos\theta+v^2t^2]^{3/2}}$$

* change variables to $\alpha = r - \frac{\cos\theta\,vt}{1-\beta^2\sin^2\theta}$ (completes the square)

$$\rightarrow \epsilon_0\int d^3r\,\vec{E}_1\times\vec{B}_2 = \frac{q_1q_2\beta(1-\beta^2)}{8\pi\epsilon_0 c}\hat{z} * \int_0^\pi d\theta\sin^3\theta * \frac{1}{(1-\beta^2)\sin^2\theta\,vt}\left[\frac{\cos\theta}{\sqrt{1-\beta^2\sin^2\theta}}+1\right]$$

odd under $\theta \rightarrow \theta - \pi/2$

$$= \frac{q_1q_2\hat{z}}{8\pi\epsilon_0c^2t}\int_0^\pi d\theta\sin\theta\left[\cancel{\frac{\cos\theta}{\sqrt{1-\beta^2\sin^2\theta}}}+1\right] = \frac{q_1q_2\hat{z}}{4\pi\epsilon_0c^2t}$$

(d) $\therefore \frac{d}{dt}\int d^3r\,\vec{\wp}_{em} = \frac{-q_1q_2\hat{z}}{4\pi\epsilon_0c^2t^2}$

* NB the 3rd Law only applies when you correctly compare the things which are exerting force on one another
* In fact the force on each charge is exerted by the fields & the total force cancels when one includes the momentum of the fields