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Solution to Problem 11.19

(a) * For a general $F(t)$ this is just a 1st order, linear, inhomogeneous ODE for $a(t)$

* $\dot{a} - \frac{1}{\tau} a = \frac{-F}{m\tau} \Rightarrow \frac{d}{dt}\left[e^{-\frac{t}{\tau}} a(t)\right] = \frac{e^{-t/\tau}}{m\tau} F(t) \Rightarrow \boxed{a(t) = a_0 e^{\frac{t}{\tau}} - \frac{1}{m\tau} e^{\frac{t}{\tau}} \int_0^t dt' e^{-t'/\tau} F(t')}$

* $a(t)$ is continuous if $F(t)$ is piece-wise smooth QED

(b) * integrating again gives

$$v(t) = v_0 + a_0\tau\left[e^{t/\tau} - 1\right] - \frac{1}{m\tau}\int_0^t dt' e^{\frac{t'}{\tau}} \int_0^{t'} dt'' e^{-\frac{t''}{\tau}} F(t'')$$

$$= v_0 + a_0\tau\left[e^{\frac{t}{\tau}} - 1\right] + \frac{1}{m}\int_0^t dt' F(t')\left[1 - e^{\frac{(t-t')}{\tau}}\right]$$

* one more integration gives

$$x(t) = x_0 + v_0 t + a_0\tau^2\left[e^{\frac{t}{\tau}} - 1 - \frac{t}{\tau}\right] + \frac{1}{m}\int_0^t dt' F(t')\left[t + \tau - t' - \tau e^{\frac{(t-t')}{\tau}}\right]$$

$\boxed{F(t) = F_0\,\theta(t)\,\theta(T-t)}$

* $-\infty < t < 0 \Rightarrow x(t) = x_0 + v_0 t + a_0\tau^2\left[e^{\frac{t}{\tau}} - 1 - \frac{t}{\tau}\right]$

* $0 < t < T \Rightarrow x(t) = x_0 + v_0 t + a_0\tau^2\left[e^{t/\tau} - 1 - t/\tau\right] + \frac{F_0}{m}\left[\frac{1}{2}t^2 + \tau t + \tau^2 - \tau^2 e^{\frac{t}{\tau}}\right]$

* $t < t < \infty \Rightarrow x(t) = x_0 + v_0 t + a_0\tau^2\left[e^{t/\tau} - 1 - \frac{t}{\tau}\right] + \frac{F_0}{m}\left[(t+\tau)T - \frac{1}{2}T^2 - \tau^2(1 - e^{-T/\tau})e^{\frac{t}{\tau}}\right]$

(c) * eliminating the "runaway" ($x(t) \sim e^{t/\tau}$) $\Rightarrow \boxed{a_0 = \frac{F_0}{m}(1 - e^{-T/\tau})}$

* eliminating "pre-acceleration" ($a(t) \neq 0$ for $t < 0$) $\Rightarrow a_0 = 0$

(d) * choosing $a_0 = \frac{F_0}{m}(1 - e^{-T/\tau})$ implies (with $v_0 = \frac{F_0\tau}{m}(1 - e^{-T/\tau})$)

* $-\infty < t < 0 \Rightarrow a(t) = \frac{F_0}{m}\left[1 - e^{-\frac{T}{\tau}}\right]e^{\frac{t}{\tau}} \Rightarrow v(t) = \frac{F_0\tau}{m}\left[1 - e^{-\frac{T}{\tau}}\right]e^{t/\tau}$

* $0 < t < T \Rightarrow a(t) = \frac{F_0}{m}\left[1 - e^{-\frac{(T-t)}{\tau}}\right] \Rightarrow v(t) = \frac{F_0\tau}{m}\left[1 + \frac{t}{\tau} - e^{-\frac{(T-t)}{\tau}}\right]$

* $T < t < \infty \Rightarrow a(t) = 0 \Rightarrow v(t) = \frac{F_0 T}{m}$

(e) $\boxed{\text{Uncharged particle has}}$

* $-\infty < t < 0 \Rightarrow a(t) = 0$ & $v(t) = 0$
* $0 < t < T \Rightarrow a(t) = \frac{F_0}{m}$ & $v(t) = \frac{F_0 t}{m}$
* $T < t < \infty \Rightarrow a(t) = 0$ & $v(t) = \frac{F_0 T}{m}$