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Solution to Problem 12.6f

* to simplify the writing I define $\vec{\epsilon} \equiv \vec{E}/c$

Transform to S'

$$* \vec{\epsilon}' = \vec{\beta}\hat{\beta}\cdot\vec{\epsilon} + \gamma\left[\vec{\epsilon} - \vec{\beta}\hat{\beta}\cdot\vec{\epsilon} + \vec{\beta}\times\vec{B}\right]$$

$$* \vec{B}' = \vec{\beta}\hat{\beta}\cdot\vec{B} + \gamma\left[\vec{B} - \vec{\beta}\hat{\beta}\cdot\vec{B} - \vec{\beta}\times\vec{\epsilon}\right]$$

$$\boxed{\frac{\vec{\beta}}{1+\beta^2} = \frac{\vec{\epsilon}\times\vec{B}}{\epsilon^2+B^2}}$$

$$* \frac{\beta}{1+\beta^2} = \frac{\|\vec{\epsilon}\times\vec{B}\|}{\epsilon^2+B^2} \equiv R \Rightarrow \beta = \frac{1}{2R}\left[1-\sqrt{1-4R^2}\right] \Rightarrow \vec{\beta} = \beta\,\frac{\vec{\epsilon}\times\vec{B}}{\|\vec{\epsilon}\times\vec{B}\|}$$

* NB $\vec{\epsilon}\times\vec{B} \perp$ to both $\vec{\epsilon}$ & $\vec{B} \Rightarrow \vec{\beta}\cdot\vec{\epsilon} = 0 = \vec{\beta}\cdot\vec{B}$

$$* (\vec{\epsilon}\times\vec{B})\times\vec{B} = -B^2\vec{\epsilon} + \epsilon\cdot B\,\vec{B}$$

$$* -(\vec{\epsilon}\times\vec{B})\times\vec{\epsilon} = \epsilon\cdot B\,\vec{\epsilon} - \epsilon^2\vec{B}$$

$$\therefore \begin{cases} \vec{\epsilon}' = \gamma\left[\left(1-\frac{\beta B^2}{\|\vec{\epsilon}\times\vec{B}\|}\right)\vec{\epsilon} + \frac{\beta\epsilon\cdot B}{\|\vec{\epsilon}\times\vec{B}\|}\vec{B}\right] = \gamma\left[1-\frac{\beta B^2}{\|\vec{\epsilon}\times\vec{B}\|}\right]\left[\vec{\epsilon} + \frac{\beta\epsilon\cdot B}{\|\vec{\epsilon}\times\vec{B}\| - \beta B^2}\vec{B}\right] \\ \vec{B}' = \gamma\left[\frac{\beta\epsilon\cdot B}{\|\vec{\epsilon}\times\vec{B}\|}\vec{\epsilon} + \left(1-\frac{\beta\epsilon^2}{\|\vec{\epsilon}\times\vec{B}\|}\right)\vec{B}\right] = \gamma\,\frac{\beta\epsilon\cdot B}{\|\vec{\epsilon}\times\vec{B}\|}\left[\vec{\epsilon} + \frac{\left(\|\vec{\epsilon}\times\vec{B}\| - \beta\epsilon^2\right)}{\beta\epsilon\cdot B}\vec{B}\right] \end{cases}$$

$$\boxed{\frac{\beta\epsilon\cdot B}{\|\vec{\epsilon}\times\vec{B}\| - \beta B^2} = \frac{\|\vec{\epsilon}\times\vec{B}\| - \beta\epsilon^2}{\beta\epsilon\cdot B}} \Rightarrow \vec{\epsilon}' \,\&\, \vec{B}' \text{ are parallel}$$

Proof

* NB $\|\vec{\epsilon}\times\vec{B}\|^2 = \epsilon^2B^2 - (\epsilon\cdot B)^2$

$$* \frac{\beta\epsilon\cdot B}{\|\vec{\epsilon}\times\vec{B}\| - \beta B^2} \stackrel{?}{=} \frac{\|\vec{\epsilon}\times\vec{B}\| - \beta\epsilon^2}{\beta\epsilon\cdot B} \Leftrightarrow \beta^2(\epsilon\cdot B)^2 \stackrel{?}{=} \|\vec{\epsilon}\times\vec{B}\|^2 - \beta\left[\epsilon^2+B^2\right]\|\vec{\epsilon}\times\vec{B}\| + \beta^2\epsilon^2B^2$$

$$\Leftrightarrow 0 \stackrel{?}{=} \|\vec{\epsilon}\times\vec{B}\|^2 - \beta\left[\epsilon^2+B^2\right]\|\vec{\epsilon}\times\vec{B}\| + \beta^2\|\vec{\epsilon}\times\vec{B}\|^2$$

$$\Leftrightarrow 0 \stackrel{?}{=} 1 - \beta/R + \beta^2 \quad \text{QED}$$

* It is not possible to find a frame for which $\vec{\epsilon}'\cdot\vec{B}' = 0$ because $\vec{\epsilon}\cdot\vec{B} \neq 0$ is an invariant (cf problem 12.5b)