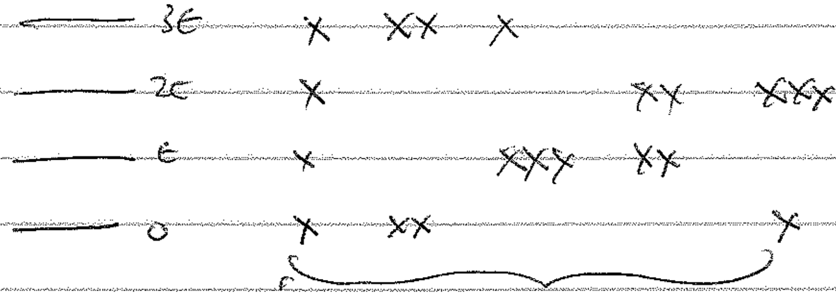
Jap. 6

2. $Z_1 = e^{-\frac{3}{2}\beta\epsilon} + 3e^{-\frac{1}{2}\beta\epsilon} + 3e^{\frac{1}{2}\beta\epsilon} + e^{\frac{3}{2}\beta\epsilon}$

$= \left(e^{-\frac{1}{2}\beta\epsilon} + e^{\frac{1}{2}\beta\epsilon}\right)^3 = 2^3\cosh^3\left(\frac{\beta\epsilon}{2}\right)$

$Z_N = (Z_1)^N \qquad F = -Nk_BT\,3\left[\ln 2 + \log\cosh\frac{\beta\epsilon}{2}\right]$

4.



F B

(1) (5)

Distinguishable: $4! = 24$ $\quad {}^4C_2 = 6$ $\quad 4$ $\quad {}^4C_2$ $\quad 4$ $\Rightarrow \Sigma = 44$.

6. $W = {}^N C_n \cdot 3^n = \frac{N!}{n!\,(N-n)!}\,3^n$

arrange spins

$S = k_B \ln W = n\ln 3 + N\ln N - n\ln N - (N-n)\ln(N-n)$

$= N\left\{\frac{n}{N}\ln 3 + \ln N - \frac{n}{N}\ln N - \left(1-\frac{n}{N}\right)\ln(N-n)\right\}$

$= N\left\{\frac{n}{N}\ln 3 - \left(1-\frac{n}{N}\right)\ln\left(1-\frac{n}{N}\right) - \frac{n}{N}\ln\frac{n}{N}\right\}$

$= N\left\{x\ln 3 - (1-x)\ln(1-x) - x\ln x\right\}$

6.

$$S = \frac{\partial U}{\partial T} = \left(\frac{\partial U}{\partial n}\right)\left(\frac{\partial n}{\partial T}\right) \qquad U = n\epsilon = Nx\epsilon$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial U}\right)_V = \left(\frac{\partial S}{\partial x}\right)\left(\frac{\partial x}{\partial U}\right) \qquad x = \frac{n}{N}.$$

$$\frac{\partial S}{\partial x} = k_B N\{\ln 3 - \ln x + \ln(1-x)\} \qquad \frac{\partial U}{\partial x} = N\epsilon$$

$$\frac{1}{T} = \frac{1}{\epsilon} k_B \ln\left(\frac{3-3x}{x}\right)$$

$$\frac{3-3x}{x} = e^{\epsilon/k_BT}$$

$$\frac{3}{x} - 3 = e^{\epsilon/k_BT} \qquad \frac{3}{x} = 3 + e^{\epsilon/k_BT}$$

$$x = \frac{3}{3+e^{\epsilon/k_BT}}$$

$$\frac{n}{N} = \frac{3}{3+e^{\epsilon/k_BT}}$$

7.

$$Z^1 = e^{-\beta\mu B} + e^{\beta\mu B}$$

$$Z^N = (Z^1)^N$$

$$F = -k_B T N \ln Z^1 = -k_B T N \ln\left(e^{-\beta\mu B} + e^{\beta\mu B}\right)$$

$$= -N k_B T \left\{ \beta\mu B + \ln\left(1 + e^{-2\beta\mu B}\right) \right\}$$

$$= -N \left\{ \mu B + k_B T \ln\left(\quad \right) \right\}$$

$$S = -\left(\frac{\partial F}{\partial T}\right)_V = N \left\{ k_B \ln\left(1 + e^{-2\beta\mu B}\right) + k_B T \frac{1}{1 + e^{-2\beta\mu B}} e^{-2\beta\mu B} \left(\frac{2\mu B}{k_B T^2}\right) \right\}$$

$$= N k_B \left\{ \ln\left(1 + e^{-2\beta\mu B}\right) + \frac{2\mu B}{k_B T} \left(\frac{e^{-2\beta\mu B}}{1 + e^{-2\beta\mu B}} \right) \right\}$$

$$S = F\left(\frac{\mu B}{k_B T}\right)$$

Isentropic Process $\quad S = \text{constant}$

$$\left(\frac{B}{T}\right)_{INIT} = \left(\frac{B}{T}\right)_{final}$$

$$\frac{10}{1} = \frac{10^{-2}}{T_{Final}} \qquad T_{Final} = 1\,\mathrm{mK}$$

9.

$$Z_{NI} = \frac{z_a^{N_a}\, z_b^{N_b}}{N_a!\, N_b!} \qquad N_a + N_b = N$$

$$F_{NI} = -k_BT \ln Z_{NI} = N_a \ln z_a + N_b \ln z_b - N_a \ln N_a - N_b \ln N_b$$

$$Z_I = \frac{z_a^N}{N!}$$

$$F_I = -k_BT = N \ln z_a - N \ln N.$$

$$= N \ln z_a - (N_a + N_b) \ln (N_a + N_b)$$

$$\Delta F = k_BT \{ -N_a \ln N_a - N_b \ln N_b + (N_a + N_b) \ln (N_a + N_b) \}$$

$$S = -\frac{\partial}{\partial T}(\Delta F) = k_B \{ N_a \ln N_a + N_b \ln N_b - N_a \ln N - N_b \ln N \}$$

$$= k_B \left\{ N_a \ln \frac{N_a}{N} + N_b \ln \frac{N_b}{N} \right\}$$

$\longrightarrow$

Chap. 7

3. $\epsilon(k) = \alpha k^{3/2}$ $\qquad \frac{\partial \epsilon}{\partial k} = \frac{3}{2}\alpha k^{1/2}$ $\qquad k^2$

$$\overset{3D}{D}(\cancel{\epsilon}) = \frac{V}{2\pi^2}\frac{k^2}{\partial\epsilon/\partial k} = \frac{V}{2\pi^2}\cdot\frac{k^2}{3/2\,\alpha k^{1/2}}$$

$$= \frac{V}{3\pi^2\alpha} k^{3/2}$$

$$D(\epsilon) = \frac{V}{3\pi^2\alpha^2}\cdot\epsilon$$

$\longrightarrow$

6.

$$\epsilon^2 = p^2c^2 + m^2c^4 \simeq p^2c^2 = (\hbar k)^2c^2$$

$$\epsilon = \hbar k c \qquad \frac{\partial\epsilon}{\partial k} = \hbar c \qquad k^2 = \frac{\epsilon^2}{\hbar^2c^2}$$

$$D(\cancel{\epsilon}) = \frac{V}{2\pi^2}\frac{k^2}{\partial\epsilon/\partial k} = \frac{V}{2\pi^2}\frac{k^2}{\hbar c}.$$

$$D(\epsilon) = \frac{V}{2\pi^2}\cdot\frac{\epsilon^2}{(\hbar c)^3}$$

$$Z = \int e^{-\beta\epsilon}\, d\epsilon\, D(\epsilon)$$

$$= \int e^{-\beta\epsilon}\cdot\frac{V}{2\pi^2}\frac{\beta^2\epsilon^2}{(\hbar c)^3}\beta d\epsilon \cdot \beta^{-3}.$$

6/

$$Z' = \frac{V}{2\pi^2}\left(\frac{k_BT}{\hbar c}\right)^3 \underbrace{\left(\int e^{-x} x^2 dx\right)}_{=2}$$

$$Z_N = (Z')^N / N!$$

$$= \left[\frac{V(k_BT)^3}{\pi^2(\hbar c)^3}\right]^N / N!$$

11.

$$D(\epsilon) = \frac{V}{2\pi^2}\frac{k^2}{\partial\epsilon/\partial k} \qquad \epsilon = \alpha k^3$$

$$\partial\epsilon/\partial k = 3\alpha k^2$$

$$= \frac{V}{6\pi^2\alpha}$$

$$D(\epsilon) = \frac{V}{6\pi^2\alpha}$$

$$Z = \int e^{-\beta\epsilon} D(\epsilon)\, d\epsilon$$

$$= \frac{V}{6\pi^2\alpha}\int e^{-\beta\epsilon}(d\epsilon\beta)\,\beta^{-1}$$

$$= \frac{V k_BT}{6\pi^2\alpha}$$

17 In 3D, density of states

$$D(k) = \frac{V}{2\pi^2} k^2.$$

$p = mu = \hbar k$

$k = \frac{mu}{\hbar}$

Number density

$$n(u) = \cancel{\int} \frac{V}{2\pi^2} \frac{k^2}{Z} e^{-\frac{mu^2}{2kT}} \cancel{dk}.$$

$$= \frac{V}{2\pi^2} \left(\frac{m}{\hbar}\right)^3 \cancel{\int} \frac{u^2}{Z} e^{-\frac{mu^2}{2kT}} \cancel{du}.$$

Dist^n of speeds at hole

$$u\,n(u) = \frac{V}{2\pi^2} \left(\frac{m}{\hbar}\right)^3 \frac{1}{Z} u^3 e^{-\frac{1}{2}mu^2/kT}.$$

av. k.e. leaving

$$\left(\tfrac{1}{2}mu^2\right) = \int \left(\tfrac{1}{2}mu^2\right) u\,n(u)\,du \Big/ \int u\,n(u)\,du$$

$$= \frac{m}{2} \int u^5 e^{-\frac{1}{2}mu^2/kT} \Big/ \int u^3 e^{-\frac{1}{2}mu^2/kT}$$

$$= k_B T \int x^5 e^{-x^2} \Big/ x^3 e^{-x^2}$$

$$= 2k_B T$$

Chap. 8

1. $$\bar{U} = \int \mathcal{D}^{(2)}(k)\,dk \; \frac{\hbar\omega}{e^{\beta\hbar\omega}-1} \qquad \omega = \left(\frac{\gamma_s}{\rho}\right)^{1/2} k^{3/2}$$

$$= k_B T \int \frac{A}{2\pi} k\,dk \cdot \frac{\beta\hbar\omega}{e^{\beta\hbar\omega}-1} \qquad \beta\hbar\omega = \beta\hbar\left(\frac{\gamma_s}{\rho}\right)^{1/2} k^{3/2} = z^{3/2}$$

$$k^{3/2} = z^{3/2}\left(\frac{\rho}{\gamma_s}\right)^{1/2}\frac{1}{\beta\hbar}$$

$$= k_B T \frac{A}{2\pi}\int \left(\frac{\rho}{\gamma_s}\right)^{2/3}\left(\frac{1}{\beta\hbar}\right)^{4/3} z\,dz \; \frac{z^{3/2}}{e^{z^{3/2}}-1} \qquad k = z\left(\frac{\rho}{\gamma_s}\right)^{1/3}\left(\frac{1}{\beta\hbar}\right)^{2/3}$$

$$k\,dk = z\,dz\left(\frac{\rho}{\gamma_s}\right)^{2/3}\left(\frac{1}{\beta\hbar}\right)^{4/3}$$

$$\bar{U} = k_B T \frac{A}{2\pi}\left(\frac{\rho}{\gamma_s}\right)^{2/3}\left(\frac{kT}{\hbar}\right)^{4/3}\int_0^\infty \frac{z^{5/2}\,dz}{e^{z^{3/2}}-1}$$

4. $$\langle E\rangle = U = -\frac{1}{Z}\frac{\partial Z}{\partial\beta} \qquad \langle E^2\rangle = \frac{1}{Z}\frac{\partial^2 Z}{\partial\beta^2}$$

$$\Delta E^2 = \langle E^2\rangle - \langle E\rangle^2 = \frac{\partial^2 \ln Z}{\partial\beta^2} = -\left(\frac{\partial U}{\partial\beta}\right)_{V,N}$$

$$\overline{(\Delta E)^2} = -\left(\frac{\partial U}{\partial T}\right)_{V,N}\left(\frac{\partial T}{\partial\beta}\right)_{V,N} = k_B T^2 C_V$$

Photons: $$\bar{U} = \frac{\pi^2 k_B^4 T^4}{15\hbar^3 c^3} V$$

$$\Delta E^2 = kT^2 C_V = kT^2\frac{\partial U}{\partial T} = \frac{4\pi^2 (k_B T)^5}{15\hbar^3 c^3} V.$$

6. $$U = \int_0^{\omega_D} \mathcal{D}(\omega)\, d\omega \cdot \hbar\omega \cdot \bar{n}(\omega)$$

$$= \int_0^{\omega_D} V \frac{\omega^2 d\omega}{2\pi \bar{s}^3} \frac{\hbar\omega}{e^{\beta\hbar\omega}-1}$$

Use $x = \beta\hbar\omega$

$$U = \frac{V\hbar}{2\pi\bar{s}^3} \frac{1}{(\beta\hbar)^4} \int_0^{\omega_D} \frac{x^3 dx}{e^x - 1}$$

Integrand,

$$I = \frac{x^3}{e^x - 1}$$

Maximum at $\frac{\partial I}{\partial x} = 0 = \frac{3x^2}{e^x - 1} - \frac{x^3 e^x}{(e^x - 1)^2}$

Solution: $3e^x - 3 = xe^x$

$$x = 3(1 - e^{-x})$$

Solve by iteration:

1st approx $x = 3$

2nd approx $x = 3(1 - e^{-3}) = 2.85$

6 contd.

$$x = \beta \hbar \omega = 2.85$$

$$\text{Energy} = \hbar \omega = 2.85\, kT$$

At 30K $\quad E_x = 2.85 * 1.4 * 10^{-23} \times 30$

$$= 1.1 * 10^{-21}\ J.$$

At 625K $\quad E_x = \frac{625}{30} \times 1.1 \times 10^{-21}\ J$

$$= 2.5 \times 10^{-20}\ J.$$

12. Stefan-Boltzmann

$$\dot{Q} = A\sigma T^4.$$

$$\dot{U} = -A\sigma T^4 \qquad U = Mc^2$$

$$\dot{U} = \dot{M}c^2$$

$$\dot{M}c^2 = -A\sigma T^4$$

$$= -4\pi\left(\frac{2GM}{c^2}\right)^2 \sigma T^4.$$

$$= -4\pi \frac{4G^2M^2}{c^4} \sigma T^4.$$

$$= -16\pi \frac{G^2M^2}{c^4} \sigma \left(\frac{\hbar c^3}{\pi k_B G}\right)^4 M^{-4}.$$

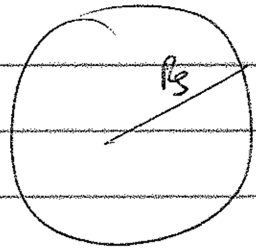
$$\dot{M}c^2 = -BM^{-2} \qquad B = \text{constant}$$

$$M^2\dot{M} = -\frac{B}{c^2} \qquad = \frac{16 G^{-2} M^{-2} \sigma c^{\cancel{8}5}}{\pi^3 \cancel{c^3} k_B^4}$$

$$\frac{1}{3}M^3 = -\frac{B}{c^2}t$$

Characteristic time $\tau \simeq \frac{c^2}{3B} \approx \frac{\pi^3 \cancel{c^3} k_B^4 c^{-5}}{16 G^{-2} M^2 \sigma \cancel{c^8}}$

12



$$A = 4\pi R_s^2 \qquad R_s = \frac{2GM}{c^2}.$$

$$S = \frac{k_B c^3 A}{4G\hbar} = \frac{k_B}{\cancel{4}} \frac{\cancel{c^3} \cdot \cancel{4}\pi}{\cancel{G}\hbar c} \frac{4G^{\cancel{2}}M^2}{\cancel{c^4}}$$

$$= \frac{4\pi k_B G}{\hbar c} M^2 \qquad E = Mc^2$$

$$M^2 = \frac{E^2}{c^4}$$

$$= \frac{4\pi k_B G}{\hbar c} \frac{E^2}{c^4}$$

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_V = \frac{8\pi k_B G E}{\hbar c^5} = \frac{8\pi k_B G M}{\hbar c^3}.$$

Stefan-Boltzmann

$$\dot{U} = -A\sigma T^4 \qquad U = Mc^2$$

$$\dot{M}c^2 = -4\pi \left(\frac{2GM}{c^2}\right)^2 \sigma T^4$$

$$= -16\pi \frac{G^2M^2}{c^4} \sigma T^4$$

Use T from above

12 contd.

$$\dot{M}c^2 = -16\pi \frac{G^2M^2}{c^4}\sigma \Big/ \left(\frac{8\pi k_B G M}{\hbar c^3}\right)^4$$

$$= -16 \frac{G^2M^2\sigma}{c^4} \frac{(\hbar c^3)^4}{(8\pi k_B)^4} \frac{1}{G^4M^4}.$$

$$M^2\dot{M} = \frac{-16\pi\sigma}{c^6} \frac{\hbar^4 c^{12}}{(64)^2\pi^4 k_B^4} \frac{1}{G^2}$$

$$\frac{d}{dt}(M^3) = -\frac{48\sigma}{(64)^2} \frac{\hbar^4 c^6}{\pi^3 k_B^4} \frac{1}{G^2}$$

$$M^3(t) - M^3(0) = -\left[\frac{3}{256}\left(\frac{\hbar c}{k_B}\right)^4 \frac{c^2\sigma}{\pi^3 G^2}\right] t$$

$M^3(t) = 0$ at $t = \tau$

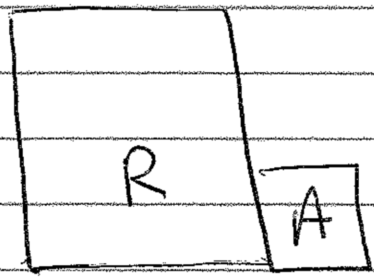
for $$\tau = \frac{M(0)^3}{3} \Big/ \left(\frac{\hbar c}{4k_B}\right)^4 \frac{c^2\sigma}{\pi^3 G^2}$$

$$\left(\frac{\hbar c}{4k_B}\right)^4 \frac{c^2\sigma}{\pi^3G^2} \simeq 4\times 10^{15}. \qquad M^3(0) = 8\times 10^{33}$$

$$\tau \simeq \frac{8\times 10^{33}}{3} \frac{1}{4\times 10^{15}} \simeq 6.7\times 10^{17}\ \text{s.}$$

Hmwk 9

#2.



Reservoir

$$\frac{1}{T} = k_B \frac{\partial}{\partial U_R}\left(\ln W_R\right)_{N,L} = \frac{\partial S_R}{\partial U_R}$$

$$w_z = -k_B T \frac{\partial}{\partial N_R}\left(\ln W_R\right) = -T\frac{\partial S_R}{\partial L_z}$$

Solve above

$$\ln W_R = \text{const} + \frac{U_R - w_z L_{zR}}{k_B T}$$

$$W_R\left(U_R, L_{zR}\right) = \gamma \exp\left(\frac{U_R - w_z L_{zR}}{k_B T}\right).$$

For quantum state $|\epsilon_i\rangle$

$$W(\epsilon_i) = 1 * W\left(U_R - w_z L_{zR}\right)$$

$$= 1 * W_R\left[\left(U_T - U_A\right) - w_z\left(L_{z_T} - L_{z_A}\right)\right]$$

$$= \text{const} * \exp\left[-\left(U_A - w_z L_{z_A}\right)/k_B T\right]$$

Prob. for state i

$$P_i = \frac{\exp\left[-\left(U_A - w_z L_{z_A}\right)(k_B T)\right]}{\sum_{A_i} \exp(\quad)} . \leftarrow \quad \mathcal{N} = \sum_i e^{-\beta(\epsilon_i - wL}$$

#2.

$$S = -k_B \sum_i p_{z_i} \ln p_{z_i}$$

$$= k_B \left\{ \sum_i p_{z_i} \frac{E_{z_i} - \omega_z L_{z_i}}{k_B T} + \ln \mathcal{X} \right\}$$

$$= k_B \ln \mathcal{X} + \frac{1}{T}\left(\bar{U} - \omega_z \bar{L}_z\right).$$

$$ST = k_B T \ln \mathcal{X} + \bar{U} - \omega_z \bar{L}_z$$

or

$$\bar{U} - ST - \omega_z \bar{L}_z = -k_B T \ln \mathcal{X} = \Phi_G$$

6.

$$\vec{F} = mr\omega^2 \hat{r} = -\vec{\nabla} V$$

$$V = -\tfrac{1}{2} m\omega^2 r^2$$

$$n(r) = n_o e^{-\beta V} = n_o e^{\frac{m\omega^2 r^2}{2k_B T}}$$

$$PV = nRT$$

$$\therefore P = P_o e^{\frac{m\omega^2 r^2}{2k_B T}}$$

$n = n_o e^{-\beta V}$ valid if $V(r)$ slowly varying with r c.f to mean separation of particles (averaged over time scale ~10× collision time.

8. Φ_G fn of V, N, T

$$d\Phi_G = \underbrace{\left(\frac{\partial \Phi_G}{\partial V}\right)_{\mu,T}}_{-P} dV + \underbrace{\left(\frac{\partial \Phi_G}{\partial \mu}\right)_{V,T}}_{-\bar{N}} d\mu + \underbrace{\left(\frac{\partial \Phi_G}{\partial T}\right)_{\mu,V}}_{-S} dT.$$

If μ, T are constant.

$$\int d\Phi_G = -PV + \text{constant.}$$

$$\Phi_G \propto P \qquad \therefore \left(\frac{\partial \Phi_G}{\partial V}\right)_{\mu,T} = -P = \frac{\Phi_G}{V}.$$

$$\Phi_G = -PV$$

$$d\Phi_G = -\cancel{PdV} - VdP = -\cancel{PdV} - \bar{N}d\mu - SdT$$

$$VdP - \bar{N}d\mu - SdT = 0.$$

9. $U = N\epsilon$

possibilities

$$W = \frac{M!}{N!\,(M-N)!}$$

Grand Canonical Partition Function

$$Z = \sum_i W(M,N)\, e^{-\beta(U - N\mu)}$$

$$= \sum_i \frac{M!}{N!\,(M-N)!}\, e^{-\beta N(\epsilon-\mu)}$$

$$= \left[1 + e^{-\beta(\epsilon-\mu)}\right]^M$$