Angular Momentum Commutators:

$\vec{L} = \vec{r} \times \vec{p}$

$$L_x = y p_z - z p_y$$
$$L_y = z p_x - x p_z$$
$$L_z = x p_y - y p_x$$

$$L_x^\dagger = p_z^\dagger y^\dagger - p_y^\dagger z^\dagger = p_z y - p_y z = y p_z - z p_y = L_x,$$

i.e., $\vec{L}$ is hermitian.

Why is $\vec{L}$ important?

1) Observable
2) $[H, \vec{L}] = 0$ for $H = \frac{p^2}{2m} + V(|r|)$, $|r| = \sqrt{x^2 + y^2 + z^2}$

$\rightarrow$ simultaneous eigenvectors

3) Relation to rotations

Basic commutator:

$$[L_x, L_y] = [y p_z - z p_y, z p_x - x p_z]$$

$$\begin{aligned}
&= [y p_z, z p_x] \leftsquigarrow y p_x [p_z, z] = -i\hbar\, y p_x \\
&+ [z p_y, x p_z] \leftsquigarrow p_y x [z, p_z] = i\hbar\, p_y x \\
&- [z p_y, z p_x] \leftsquigarrow 0 \\
&- [y p_z, x p_z] \leftsquigarrow 0
\end{aligned}$$

$$\begin{aligned}
&= i\hbar (x p_y - y p_x) \\
&= i\hbar L_z
\end{aligned}$$

$$[L_x, L_y] = i\hbar L_z$$

$$[L_y, L_z] = i\hbar L_x \qquad \text{(similarly)}$$

$$[L_z, L_x] = i\hbar L_y \qquad \text{(similarly)}$$

Implications of: $\boxed{\begin{aligned}[J_x, J_y] &= i\hbar J_z \\ [J_y, J_z] &= i\hbar J_x \\ [J_z, J_x] &= i\hbar J_y\end{aligned}}$

Let $\boxed{\begin{aligned}J_+ &= J_x + iJ_y \\ J_- &= J_x - iJ_y \\ J^2 &= J_x^2 + J_y^2 + J_z^2\end{aligned}}$ $\boxed{[J^2, J_\alpha] = 0 \text{ for } \alpha = x, y, z}$ ★

$$[J_+, J_z] = [J_x + iJ_y, J_z] = [J_x, J_z] + i[J_y, J_z]$$
$$= -i\hbar J_y + i(i\hbar J_x)$$
$$= -\hbar J_+, \text{ i.e.,}$$

$$\boxed{[J_z, J_+] = \hbar J_+}$$

$$[J_z, J_-] = [J_z, J_x - iJ_y] = [J_z, J_x] - i[J_z, J_y]$$
$$= i\hbar J_y - i(-i\hbar J_x)$$
$$= -\hbar J_x + i\hbar J_y$$
$$= -\hbar(J_x - iJ_y)$$
$$\boxed{[J_z, J_-] = -\hbar J_-}$$

$$[J_+, J_-] = [J_x + iJ_y, J_x - iJ_y]$$
$$= [J_x, -iJ_y] + [iJ_y, J_x]$$
$$= -i(i\hbar J_z) + i(-i\hbar J_z)$$
$$\boxed{[J_+, J_-] = 2\hbar J_z}$$

★ $[J^2, J_x] = [J_y^2, J_x] + [J_z^2, J_x] = J_y^2 J_x - J_y J_x J_y + J_y J_x J_y - J_x J_y^2$
$J_z^2 J_x - J_z J_x J_z + J_z J_x J_z - J_x J_z^2$
$= J_y[J_y, J_x] + [J_y, J_x]J_y + J_z[J_z, J_x] + [J_z, J_x]J_z$

$$= J_y(-i\hbar J_z) + (-i\hbar J_z)J_y + J_z(i\hbar J_y) + (i\hbar J_y)J_z$$
$$= 0$$

$$
\begin{aligned}
J_+ J_- &= (J_x + iJ_y)(J_x - iJ_y) \\
&= J_x^2 + J_y^2 + i[J_y, J_x] \\
&= J_x^2 + J_y^2 + i(-i\hbar J_z) \\
&= J_x^2 + J_y^2 + \hbar J_z
\end{aligned}
$$

$$\boxed{J_+ J_- = J^2 - J_z^2 + \hbar J_z}$$

$$
\begin{aligned}
J_- J_+ &= (J_x - iJ_y)(J_x + iJ_y) \\
&= J_x^2 + J_y^2 - i[J_y, J_x] \\
&= J_x^2 + J_y^2 - i(-i\hbar J_z) \\
&= J_x^2 + J_y^2 - \hbar J_z
\end{aligned}
$$

$$\boxed{J_- J_+ = J^2 - J_z^2 - \hbar J_z}$$

$$\frac{1}{2}(J_+ J_- + J_- J_+) = J^2 - J_z^2$$

$$\boxed{J^2 = \frac{1}{2}(J_+ J_- + J_- J_+) + J_z^2}$$