Eigenvectors of Angular Momentum:

We know that $J_+|j,m\rangle \propto |j,m+1\rangle$
$$J_-|j,m\rangle \propto |j,m-1\rangle$$

and
$$\langle j,m|J_-J_+|j,m\rangle = \hbar^2(\lambda - \lambda_z^2 - \lambda_z)$$
$$= \hbar^2(j(j+1) - m(m+1))$$
$$\langle j,m|J_+J_-|j,m\rangle = \hbar^2(\lambda - \lambda_z^2 + \lambda_z)$$
$$= \hbar^2(j(j+1) - m(m-1)).$$

Thus, the prefactor is taken to be

$$\boxed{\begin{aligned} J_+|j,m\rangle &= \hbar\sqrt{j(j+1)-m(m+1)}\;|j,m+1\rangle \\ J_-|j,m\rangle &= \hbar\sqrt{j(j+1)-m(m-1)}\;|j,m-1\rangle. \end{aligned}}$$

$\Rightarrow$ If we know one of the $|j,m\rangle$ we can find the others for the same j using J_+ and J_-.

For orbital angular momentum

$$L_{\pm} = \pm\hbar\, e^{\pm i\varphi}\left(\frac{\partial}{\partial\theta} \pm i\cot\theta\,\frac{\partial}{\partial\varphi}\right)$$

$$L_z = \frac{\hbar}{i}\frac{\partial}{\partial\varphi}\,.$$

Look for solutions

$$L_z Y_{\ell m} = \hbar m Y_{\ell m}$$

$$L^2 Y_{\ell m} = \hbar^2 \ell(\ell+1) Y_{\ell m}\,.$$

$$\frac{\hbar}{i}\frac{\partial}{\partial\varphi} Y_{\ell m} = \hbar m Y_{\ell m} \longrightarrow \boxed{Y_{\ell m}(\theta,\varphi) = f(\theta)e^{im\varphi}}$$

Rather than solve using L^2, use

$$L_+ Y_{\ell\ell} = 0 = \hbar e^{i\varphi}\left(\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\varphi}\right) f(\theta)e^{i\ell\varphi}$$

$$= \hbar e^{i\varphi}\left(\frac{df}{d\theta} + i\cot\theta\,(i\ell)\,f\right)e^{i\ell\varphi}$$

$$\longrightarrow \quad \frac{df}{d\theta} = \ell\cot\theta\, f = \ell\,\frac{\cos\theta}{\sin\theta}\, f$$

$\rightarrow f(\theta) \propto \sin^{\ell}(\theta)$

check: $\frac{d}{d\theta} \sin^{\ell}(\theta) = \ell \sin^{\ell-1}(\theta) \cos\theta$
$= \ell \frac{\cos\theta}{\sin\theta} \sin^{\ell}(\theta).$

Thus, $Y_{\ell\ell}(\theta,\varphi) = C \sin^{\ell}(\theta) e^{i\ell\varphi}$

To get the normalization constant C:

$$1 = |C|^2 \int d\Omega \, \sin^{2\ell}(\theta)$$

$$= |C|^2 \int_0^{2\pi} d\varphi \int_0^{\pi} d\theta \, \sin\theta \, \sin^{2\ell}(\theta)$$

$$= |C|^2 \, 2\pi \int_{-1}^{1} du \, (1-u^2)^{\ell} \quad \text{, where } u = \cos\theta, \ \sin^2\theta = 1-u^2$$

$$= |C|^2 \, 4\pi \int_0^1 du \, (1-u^2)^{\ell}$$

$$= |C|^2 \, 4\pi \, \frac{2^{2\ell}(\ell!)^2}{(2\ell+1)!}$$ (see next page for details)

$$\rightarrow |C| = \frac{1}{2^{\ell}\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}}$$

The sign or phase convention is

$$\boxed{Y_{\ell\ell}(\theta,\varphi) = \frac{(-1)^{\ell}}{2^{\ell}\ell!}\sqrt{\frac{(2\ell+1)!}{4\pi}} \, (\sin\theta)^{\ell} e^{i\ell\varphi}.}$$

Details on evaluating integral:

Integrate by parts:

$$\mathcal{I}_\ell = \int_0^1 du\, \underbrace{(1-u^2)^\ell}_{v} = \underbrace{u(1-u^2)^\ell}_{uv}\Big|_0^1 - \int_0^1 du\, u\,\ell(1-u^2)^{\ell-1}(-2u)$$

$$= 2\ell \int_0^1 du\, u^2 (1-u^2)^{\ell-1}$$

$$= 2\ell \int_0^1 du\, (u^2-1+1)(1-u^2)^{\ell-1}$$

$$= 2\ell\,(\mathcal{I}_{\ell-1} - \mathcal{I}_\ell)$$

$$\rightarrow \mathcal{I}_\ell\,(2\ell+1) = 2\ell\,\mathcal{I}_{\ell-1}$$

$$\mathcal{I}_\ell = \frac{2\ell}{2\ell+1}\mathcal{I}_{\ell-1}$$

Using $\mathcal{I}_0 = 1$, $\mathcal{I}_1 = \frac{2}{3}$, $\mathcal{I}_2 = \frac{2^2\cdot 2\cdot 1}{5\cdot 3}$,

$$\mathcal{I}_3 = \frac{2^3\cdot 3\cdot 2\cdot 1}{7\cdot 5\cdot 3\cdot 1} = \frac{2^3\,3!\;2^3\,3!}{7!}$$

and hence $\mathcal{I}_\ell = \dfrac{2^{2\ell}(\ell!)^2}{(2\ell+1)!}$