Name:

Exam 1 - PHY 4604 - Fall 2011
October 5, 2011
8:20-10:20PM, MAEA 303

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Unless otherwise noted all parts (a), (b), ... are worth 5 points, and the entire exam is 100 points.

Harmonic oscillator:

$$\begin{aligned}
a_+ &= \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x) \\
a_- &= \frac{1}{\sqrt{2\hbar m\omega}}(+ip + m\omega x) \\
x &= \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \\
p &= i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-) \\
\psi_0(x) &= \left(\frac{m\omega}{\pi\hbar}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right) \\
\psi_n &= \frac{1}{\sqrt{n!}}\,(a_+)^n\,\psi_0
\end{aligned}$$

Delta function potential $V(x) = \alpha\,\delta(x)$:

$$\frac{d\varphi(0^+)}{dx} - \frac{d\varphi(0^-)}{dx} = \frac{2m\alpha}{\hbar^2}\,\varphi(0).$$

1. **Short answer section**

(a) Write down the time dependent Schrodinger equation in one dimension.

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m} \frac{d^2\psi}{dx^2} + V\psi$$

(b) What is the physical meaning of $|\psi(x,t)|^2$?

$|\psi(x,t)|^2 dx$ is the probability of finding a particle between x & $x+dx$ at time t.

(c) Give an expression for the probability current.

$$j = \frac{\hbar}{2mi}\left(\psi^* \frac{\partial \psi}{\partial x} - \psi \frac{\partial \psi^*}{\partial x}\right)$$

(d) If the wave function at time t is given in terms of the eigenstates of the time independent Schrodinger equation as

$$\psi(x,t) = \sum_n c_n \phi_n(x) e^{-iE_n t/\hbar},$$

what are the c_n in terms of $\psi(x,0)$?

$$c_n = \int dx\, \phi_n^*(x) \psi(x,0)$$

(e) For a free particle ($V = 0$), give the wave function at time t in terms of its fourier transform at $t = 0$, $\phi(k)$, where

$$\phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \psi(x,0) e^{-ikx} dx.$$

$$\psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} dk\, \phi(k) e^{ikx} e^{-i\frac{\hbar k^2}{2m}t}$$

2. **General properties**
   At $t = 0$ the wave function of a particle in an infinite square well between $x = -1$ and $x = +1$ is (Note: $-1 < x < 1$ *not* $0 < x < 1$.)

$$\psi(x, 0) = C\,(1 - x^2). \tag{1}$$

   (a) What is the constant C so that the wave function is normalized?

$$1 = C^2 \int_{-1}^{1} (1-x^2)^2 dx = 2C^2 \int_0^1 (1-2x^2+x^4)\, dx$$

$$= 2C^2\left(1 - \frac{2}{3} + \frac{1}{5}\right) = 2C^2\left(\frac{1}{3} + \frac{1}{5}\right) = \frac{16}{15} C^2$$

$$\rightarrow C = \sqrt{\frac{15}{16}}$$

   (b) Compute the expectation values $\langle x \rangle$, $\langle x^2 \rangle$, as well as σ_x for $\psi(x, 0)$. You may use the symmetry in the problem.

$\langle x \rangle = 0$ by symmetry

$$\langle x^2 \rangle = C^2 \int_{-1}^{1} x^2 (1-x^2)^2 dx = 2C^2 \int_0^1 (x^2 - 2x^4 + x^6)\, dx$$

$$= \frac{15}{8}\left(\frac{1}{3} - \frac{2}{5} + \frac{1}{7}\right) = \frac{1}{8}\left(5 - 6 + \frac{15}{7}\right) = \frac{1}{7}$$

$$\sigma_x = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} = \sqrt{1/7}$$

(c) Compute the expectation values $\langle p\rangle$, $\langle p^2\rangle$, as well as σ_p for $\psi(x,0)$.

$\langle p\rangle = 0$ for a real ψ

$$p^2\psi(x,0) = -\hbar^2\frac{d^2}{dx^2}C(1-x^2) = 2C\hbar^2$$

$$\rightarrow \langle p^2\rangle = 2C^2\hbar^2\int_{-1}^{1}(1-x^2)\,dx$$

$$= 2C^2\hbar^2\,2\left(1-\frac{1}{3}\right) = \frac{5}{2}\hbar^2$$

$$\sigma_p = \sqrt{\langle p^2\rangle - \langle p\rangle^2} = \sqrt{\frac{5}{2}}\,\hbar$$

(d) Use the results of (b) and (c) above to check that the uncertainty principle is satisfied.

$$\sigma_x\sigma_p = \sqrt{\frac{5}{2\cdot 7}}\,\hbar > \sqrt{\frac{1}{3}}\,\hbar > \sqrt{\frac{1}{4}}\,\hbar = \frac{\hbar}{2}$$

(e) How is the expectation value $\langle xp\rangle$ related to the expectation value $\langle px\rangle$?

$$\langle xp\rangle - \langle px\rangle = i\hbar$$

3. **Harmonic oscillator**

At $t = 0$ the wave function of a particle in a harmonic oscillator potential is given by

$$\psi(x,0) = \frac{3}{5}\,\psi_1(x) + \frac{4}{5}\,i\,\psi_2(x).$$

(a) What is $\psi(x,t)$?

$$\psi(x,t) = \frac{3}{5}\psi_1(x) e^{-i\frac{3\omega}{2}t} + \frac{4}{5} i \psi_2(x) e^{-i\frac{5\omega}{2}t}$$

(b) What is the expectation value of the momentum for $\psi(x,t)$?

$$\langle p\rangle = \int dx \left(\frac{3}{5}\psi_1(x) e^{i\frac{3\omega}{2}t} - \frac{4}{5} i \psi_2(x) e^{i\frac{5\omega}{2}t}\right)$$

$$i\sqrt{\frac{\hbar m\omega}{2}}\,(a_+ - a_-)$$

$$\left(\frac{3}{5}\psi_1(x) e^{-i\frac{3\omega}{2}t} + \frac{4}{5} i \psi_2(x) e^{-i\frac{5\omega}{2}t}\right)$$

$$= i\sqrt{\frac{\hbar m\omega}{2}}\left(\frac{3}{5}\right)\left(\frac{-4}{5}i\right)\sqrt{2}\, e^{i\omega t}$$

$$- i\sqrt{\frac{\hbar m\omega}{2}}\left(\frac{4}{5}i\right)\left(\frac{3}{5}\right)\sqrt{2}\, e^{-i\omega t}$$

$$= \sqrt{\hbar m\omega}\,\frac{12}{25}\left(e^{i\omega t} + e^{-i\omega t}\right)$$

$$= \frac{24}{25}\sqrt{\hbar m\omega}\,\cos(\omega t)$$

(c) What is the expectation value of the position for $\psi(x,t)$ and how is it related to the momentum of part (b)?

$$\langle x\rangle = \int dx \left(\frac{3}{5}\psi_1(x)e^{i\frac{3\omega}{2}t} - \frac{4}{5}i\,\psi_2(x)e^{i\frac{5\omega}{2}t}\right)$$

$$\sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-)$$

$$\left(\frac{3}{5}\psi_1(x)e^{-i\frac{3\omega}{2}t} + \frac{4}{5}i\,\psi_2(x)e^{-i\frac{5\omega}{2}t}\right)$$

$$= \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{3}{5}\right)\left(-\frac{4}{5}i\right)\sqrt{2}\,e^{i\omega t}$$

$$+ \sqrt{\frac{\hbar}{2m\omega}}\left(\frac{4}{5}i\right)\left(\frac{3}{5}\right)\sqrt{2}\,e^{-i\omega t}$$

$$= \sqrt{\frac{\hbar}{m\omega}}\,\frac{12}{25}\,i\left(e^{-i\omega t} - e^{i\omega t}\right)$$

$$= \frac{24}{25}\sqrt{\frac{\hbar}{m\omega}}\,\sin(\omega t)$$

$$\frac{d\langle x\rangle}{dt} = \frac{24}{25}\sqrt{\frac{\hbar\omega}{m}}\cos(\omega t) = \frac{\langle p\rangle}{m}$$

(d) What is the expectation value of the energy for $\psi(x,t)$?

$$\langle E\rangle = \left(\frac{3}{5}\right)^2\left(\frac{3}{2}\hbar\omega\right) + \left(\frac{4}{5}\right)^2\left(\frac{5}{2}\hbar\omega\right)$$

$$= \frac{27+80}{50}\hbar\omega = \frac{107}{50}\hbar\omega$$

(e) Construct a normalized harmonic oscillator state which has an expectation value of the energy of $(3/4)\hbar\omega$. This is not related to ψ in parts (a)-(d).

Let $\psi = \alpha\psi_0 + \beta\psi_1$, where $\alpha^2 + \beta^2 = 1$

$$\langle E\rangle = \alpha^2\frac{\hbar\omega}{2} + \beta^2\frac{3\hbar\omega}{2} = \frac{3}{4}\hbar\omega$$

$$\rightarrow \alpha^2 + 3\beta^2 = \frac{3}{2}$$

$$\alpha^2 + \beta^2 = 1$$

$$2\beta^2 = \frac{1}{2} \rightarrow \beta^2 = \frac{1}{4}, \quad \beta = \frac{1}{2}$$

$$\alpha^2 = \frac{3}{4}, \quad \alpha = \frac{\sqrt{3}}{2}$$

$$\psi = \frac{\sqrt{3}}{2}\psi_0 + \frac{1}{2}\psi_1$$

4. **Piecewise constant and delta function potentials**
For this problem consider the one dimensional time independent Schrodinger question with potential $V(x)$:

$$\begin{aligned} V(x) &= V_1(x) + V_2(x) \\ V_1(x) &= 0 \text{ for } x < 0 \\ V_1(x) &= V_o \text{ for } 0 < x \text{ with } V_o > 0 \\ V_2(x) &= \alpha\delta(x). \end{aligned}$$

In other words, this is the step potential with a delta function potential at the interface. Assume that $E > V_o$.

(a) For $E > V_o$, what is the general form of the solution for $x > 0$ and for $x < 0$. Make sure to define all variables that you introduce.

$$\psi(x>0) = B e^{ik'x} + B' e^{-ik'x}, \quad k' = \sqrt{\frac{2m(E-V_o)}{\hbar^2}}$$

$$\psi(x<0) = A e^{ikx} + A' e^{-ikx}, \quad k = \sqrt{\frac{2mE}{\hbar^2}}$$

(b) What is the boundary condition at $x = 0$ expressed in terms of your wave functions from part (a)?

$$\psi(0) = A + A' = B + B'$$

$$\psi'(0^+) - \psi'(0^-) = \frac{2m\alpha}{\hbar^2}\psi(0)$$

$$\to ik'(B - B') - ik(A - A') = \frac{2m\alpha}{\hbar^2}(B + B')$$

(c) For an incoming wave coming from the left and going to the right, which of the terms in part (a) is zero?

$B' = 0$

(d) (10 points) Solve for the transmission probability for a wave coming from the left and going to the right.

$$A + A' = B$$

$$ik'B - ik(A - A') = \frac{2m\alpha}{\hbar^2} B$$

$$\hookrightarrow \quad ik(A - A') = \left(ik' - \frac{2m\alpha}{\hbar^2}\right) B$$

$$\hookrightarrow \quad A - A' = \left(\frac{k'}{k} + i\frac{2m\alpha}{k\hbar^2}\right) B$$

$$\rightarrow \quad 2A = \left(1 + \frac{k'}{k} + i\frac{2m\alpha}{k\hbar^2}\right) B$$

$$\rightarrow \quad \frac{B}{A} = \frac{1}{\frac{1}{2}\left(1 + \frac{k'}{k}\right) + i\frac{m\alpha}{k\hbar^2}}$$

$$\rightarrow \quad \left|\frac{B}{A}\right|^2 = \frac{1}{\left(\frac{1}{2}\left(1 + \frac{k'}{k}\right)\right)^2 + \left(\frac{m\alpha}{k\hbar^2}\right)^2}$$

$$\rightarrow \quad T = \frac{k'}{k}\left|\frac{B}{A}\right|^2 = \frac{k'/k}{\left(\frac{1}{2}\left(1 + \frac{k'}{k}\right)\right)^2 + \left(\frac{m\alpha}{k\hbar^2}\right)^2}$$