Solution

**Name:**

**Exam 2 - PHY 4604 - Fall 2011**
November 16, 2011
8:20-10:20PM, MAEA 303

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

All parts of the exam are worth 4 points for a total of 100 points.

$$
\begin{aligned}
Y_0^0 &= \frac{1}{2}\sqrt{\frac{1}{\pi}} \\
Y_1^1(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta \\
Y_1^0(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}}\cdot\cos\theta \\
Y_1^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta \\
Y_2^2(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{2i\varphi}\cdot\sin^2\theta \\
Y_2^1(\theta,\varphi) &= \frac{-1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{i\varphi}\cdot\sin\theta\cdot\cos\theta \\
Y_2^0(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}}\cdot(3\cos^2\theta-1) \\
Y_2^{-1}(\theta,\varphi) &= \frac{1}{2}\sqrt{\frac{15}{2\pi}}\cdot e^{-i\varphi}\cdot\sin\theta\cdot\cos\theta \\
Y_2^{-2}(\theta,\varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}}\cdot e^{-2i\varphi}\cdot\sin^2\theta
\end{aligned}
$$

1. **Short answer section**

(a) If $\langle f|g_1\rangle = 1+i$ and $\langle f|g_2\rangle = 1$, what is $\langle g_1+g_2|f\rangle$?

$$\langle g_1+g_2|f\rangle = \langle f|g_1+g_2\rangle^* = \langle f|g_1\rangle^* + \langle f|g_2\rangle^*$$
$$= 1 - i + 1 = 2 - i$$

(b) What is the completeness condition for a set of states $|\psi_n\rangle$?

$$\mathbb{1} = \sum_n |\psi_n\rangle\langle\psi_n|$$

or any $|\psi\rangle$ can be written in the form

$$|\psi\rangle = \sum_n c_n|\psi_n\rangle$$

(c) What is $(xp)^\dagger$? Is the operator $(xp)$ Hermitian?

$$(xp)^\dagger = p^\dagger x^\dagger = px$$

Since $px \neq xp$, $(xp)$ is not Hermitian.

(d) How many $m$ values are there for $j = 5/2$?

$$m = -\frac{5}{2}, -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2} \rightarrow 6 \text{ values}$$

(e) What is the energy in eV of the second excited state ($n = 3$) for the hydrogen atom?

$$-13.6\,\text{eV}/3^2 \approx -1.5\,\text{eV}$$

2. **Eigenvectors and eigenvalues**
The Hamiltonian of a two state system is represented by the matrix

$$H = \underset{\hbar\omega}{\uparrow}\begin{pmatrix} 2 & i\sqrt{5} \\ -i\sqrt{5} & -2 \end{pmatrix}.$$

(a) What are the energy eigenvalues?

$$\det\begin{pmatrix} 2-\lambda & i\sqrt{5} \\ -i\sqrt{5} & -2-\lambda \end{pmatrix} = 0 = -4+\lambda^2-5 \rightarrow \lambda = \pm 3$$

$$\boxed{E = \pm 3\hbar\omega}$$

(b) What are the normalized energy eigenvectors?

$E = 3\hbar\omega$

$$\begin{pmatrix} 2-3 & i\sqrt{5} \\ -i\sqrt{5} & -2-3 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$\rightarrow -C_1 + i\sqrt{5}\,C_2 = 0$$

$$\rightarrow \underbrace{\frac{1}{\sqrt{6}}\begin{pmatrix} i\sqrt{5} \\ 1 \end{pmatrix}}_{|3\hbar\omega\rangle}$$

$E = -3\hbar\omega$

$$\begin{pmatrix} 2+3 & i\sqrt{5} \\ -i\sqrt{5} & -2+3 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \end{pmatrix} = 0$$

$$\rightarrow -i\sqrt{5}\,C_1 + C_2 = 0$$

$$\underbrace{\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ i\sqrt{5} \end{pmatrix}}_{|-3\hbar\omega\rangle}$$

(c) At $t = 0$ the system is in state

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}.$$

What is the expectation value of the energy at $t = 0$?

$$\langle E \rangle = \frac{1}{\sqrt{2}}(1 \ -i)\hbar\omega\begin{pmatrix} 2 & i\sqrt{5} \\ -i\sqrt{5} & -2 \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

$$= \frac{\hbar\omega}{2}(1 \ -i)\begin{pmatrix} 2-\sqrt{5} \\ -i(2+\sqrt{5}) \end{pmatrix} = \frac{\hbar\omega}{2}\left(2-\sqrt{5}-(2+\sqrt{5})\right)$$

$$\boxed{= -\sqrt{5}\hbar\omega}$$

(d) What is the expectation value of the energy at time $t > 0$?

$\langle E \rangle = -\sqrt{5}\hbar\omega$ since does not change w/ time for a time independent H.

(e) What is the state of the system at time $t > 0$?

$$\frac{1}{\sqrt{6}}(-i\sqrt{5} \ \ 1)\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{i(1-\sqrt{5})}{\sqrt{12}}$$

$$\frac{1}{\sqrt{6}}(1 \ -i\sqrt{5})\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} = \frac{(1+\sqrt{5})}{\sqrt{12}}$$

$$\boxed{|\psi(t)\rangle = \frac{i(1-\sqrt{5})}{\sqrt{12}}e^{-i3\omega t}\frac{1}{\sqrt{6}}\begin{pmatrix} i\sqrt{5} \\ 1 \end{pmatrix} + \frac{(1+\sqrt{5})}{\sqrt{12}}e^{i3\omega t}\frac{1}{\sqrt{6}}\begin{pmatrix} 1 \\ i\sqrt{5} \end{pmatrix}}$$

3. **Measurements**

Three Hermitian operators have eigenvalues, $\lambda$ and eigenvectors given below

$$\sigma_x \text{ operator } : \lambda = 1 \text{ with } \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix} \text{ and } \lambda = -1 \text{ with } \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

$$\sigma_y \text{ operator } : \lambda = 1 \text{ with } \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix} \text{ and } \lambda = -1 \text{ with } \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \end{pmatrix}$$

$$\sigma_z \text{ operator } : \lambda = 1 \text{ with } \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and } \lambda = -1 \text{ with } \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

(a) At $t = 0$, the state of the system is given by

$$\begin{pmatrix} 3/5 \\ 4i/5 \end{pmatrix}.$$

If a $\sigma_y$ measurement is made at $t = 0$, what are the possible outcomes and their associated probabilities?

| outcome | probability |
|---|---|
| $+1$ | $\left\lvert \frac{1}{\sqrt{2}} (1 \;\; -i) \begin{pmatrix} 3/5 \\ 4i/5 \end{pmatrix} \right\rvert^2 = \frac{1}{2} \frac{(3+4)^2}{25} = \frac{49}{50}$ |
| $-1$ | $\left\lvert \frac{1}{\sqrt{2}} (1 \;\; i) \begin{pmatrix} 3/5 \\ 4i/5 \end{pmatrix} \right\rvert^2 = \frac{1}{2} \frac{(3-4)^2}{25} = \frac{1}{50}$ |

(b) In all of the following, assume that $\sigma_y$ was found to be $+1$. What is the state of the system immediately after the measurement?

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \end{pmatrix}$$

(c) The hamiltonian of the system is given by $H = \hbar\omega\sigma_z$. Using the result of the measurement at $t = 0$ in part (b), what is the state of the system at time $t > 0$?

$$\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t} \\ i e^{i\omega t} \end{pmatrix}$$

(d) A measurement of $\sigma_x$ is next made at time $t > 0$. What are the possible outcomes and associated probabilities?

| outcome | probability |
|---|---|
| $+1$ | $\left\| \frac{1}{\sqrt{2}}(1 \;\; 1)\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t} \\ i e^{i\omega t} \end{pmatrix}\right\|^2 = \frac{\|ie^{i\omega t} + e^{-i\omega t}\|^2}{4}$ $= \frac{1}{4}\|e^{i\pi/2}e^{i\omega t} + e^{-i\omega t}\|^2 = \frac{1}{4}\|e^{i(\omega t+\frac{\pi}{4})} + e^{-i(\omega t+\frac{\pi}{4})}\|$ $= \cos^2(\omega t + \frac{\pi}{4}) = \frac{1}{2}(1 - \sin(2\omega t))$ |
| $-1$ | $\left\| \frac{1}{\sqrt{2}}(1 \;\; -1)\frac{1}{\sqrt{2}}\begin{pmatrix} e^{-i\omega t} \\ i e^{i\omega t} \end{pmatrix}\right\|^2 = \sin^2(\omega t + \frac{\pi}{4})$ $= \frac{1}{2}(1 + \sin(2\omega t))$ |

(e) In the answer for part (d) are there any times, $t$, for which the outcome is known with 100% certainty? If so, how can one explain 100% certainty in quantum mechanics?

Yes, when $|\psi(t)\rangle$ is an eigenstate of $\sigma_x$.

4. **Angular Momentum**

(a) Compute the commutator $[L_x, L_y L_z]$ .

$$L_x L_y L_z - L_y L_z L_x$$
$$- L_y L_x L_z + L_y L_x L_z$$
$$= [L_x, L_y] L_z + L_y [L_x, L_z]$$
$$= i\hbar L_z^2 - i\hbar L_y^2$$

(b) Compute the matrix element $\langle 2, 1|L_x|2, 0\rangle$.

$$L_x = \frac{1}{2}(L_+ + L_-)$$

Only $L_+$ links $|2,0\rangle$ to $|2,1\rangle$ so

$$\langle 2,1|L_x|2,0\rangle = \frac{1}{2}\langle 2,1|L_+|2,0\rangle$$
$$= \frac{1}{2}\hbar\sqrt{2(2+1) - 0(0+1)}$$
$$= \frac{\sqrt{6}}{2}\hbar = \sqrt{\frac{3}{2}}\hbar$$

(c) Compute the matrix element $\langle 2, 0|L_x^2|2, 0\rangle$.

$$\langle 2,0|L_x^2|2,0\rangle = \frac{1}{4}\langle 2,0|(L_+ + L_-)^2|2,0\rangle$$
$$= \frac{1}{4}\langle 2,0|L_+L_- + L_-L_+|2,0\rangle$$
$$\left.\begin{aligned} &= \frac{\hbar^2}{4}\underbrace{\sqrt{2(2+1)-0\cdot(0+1)}}_{L_+ \text{ first}}\underbrace{\sqrt{2(2+1)-1(1-1)}}_{L_- \text{ second}} \\ &+ \frac{\hbar^2}{4}\underbrace{\sqrt{2(2+1)-0(0-1)}}_{L_- \text{ first}}\underbrace{\sqrt{2(2+1)--1(1+-1)}}_{L_+ \text{ second}} \end{aligned}\right\} = \frac{2\hbar^2}{4}(\sqrt{6})^2 = 3\hbar^2$$

(d) An $L^2$ measurement is performed on a wavefunction $\psi(r,\theta,\phi) = f(r)\sin^2(\theta)$. What are the possible outcomes?

$$\sin^2\theta = 1 - \cos^2\theta = \underbrace{3 - \cos^2\theta}_{\propto Y_{2,0}} \underbrace{-2}_{\propto Y_{0,0}}$$

Outcomes: $\underbrace{\hbar^2 2(2+1)}_{6\hbar^2}, \underbrace{\hbar^2 \cdot 0}_{0}$

(e) An $L_z$ measurement is performed on a wavefunction $\psi(r,\theta,\phi) = f(r)\sin^2(\theta)$. What are the possible outcomes?

Outcome: $0$, since $e^{i0\cdot\varphi}$.

5. **Radial Schrodinger Equation**
In this problem use the potential in spherical coordinates

$$\begin{aligned} V(r) &= \infty \text{ for } r < a \\ V(r) &= -V_0 \text{ for } a < r < b \\ V(r) &= 0 \text{ for } b < r. \end{aligned}$$

with $V_0 > 0$. We are going to look for the bound states for $-V_0 < E < 0$.

(a) Solve the radial Schrodinger equation for $\ell = 0$ $u$ in the region $a < r < b$. Given that $u(a) = 0$ because the potential is infinity for $r < a$, what is the general form of the solution in this region?

$$-\frac{\hbar^2}{2m}\frac{d^2u}{dr^2} + \frac{\hbar^2 0(0+1)}{2mr^2}u - V_0 u = Eu$$

$$\rightarrow \frac{d^2u}{dr^2} = -\frac{2m(E+V_0)}{\hbar^2}u = -k^2 u$$

$$\boxed{, k = \sqrt{\frac{2m(E+V_0)}{\hbar^2}} \quad u(r) = A\sin(k(r-a))}$$

(b) Solve the radial Schrodinger equation for $\ell = 0$ $u$ in the region $r > b$. Given that we want the wave function to be normalizable, what is the general form of the solution in this region?

$$\frac{d^2u}{dr^2} = \frac{-2mE}{\hbar^2}u = K^2 u$$

$$\boxed{, K = \sqrt{\frac{2m(-E)}{\hbar^2}} \quad u(r) = Be^{-Kr}}$$

(c) Match the boundary conditions at $r = b$ and derive a equation for the bound states.

$$u(b) = A\sin(k(b-a)) = Be^{-\kappa b}$$

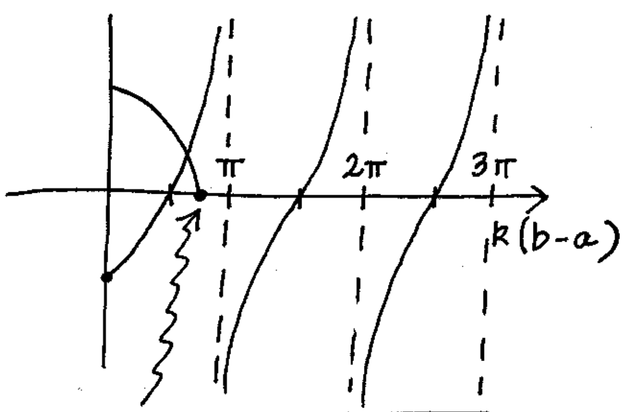
$$u'(b) = kA\cos(k(b-a)) = -\kappa B e^{-\kappa b}$$

$$\rightarrow \boxed{k\cot(k(b-a)) = -\kappa}$$

Since $k^2 + \kappa^2 = \frac{2mV_0}{\hbar^2}$, $\sqrt{\left(\frac{2mV_0}{\hbar^2}\right) - k^2} = -k\cot(k(b-a))$

and $\boxed{\sqrt{\left(\frac{2mV_0(b-a)^2}{\hbar^2}\right) - (k(b-a))^2} = -k(b-a)\cot(k(b-a))}$

(d) Derive the condition on $V_0$ for there will not be a solution (bound state) for $-V_0 < E < 0$.




$$k(b-a) = \sqrt{\frac{2mV_0(b-a)^2}{\hbar^2}}$$

no solution:

$$\sqrt{\frac{2mV_0(b-a)^2}{\hbar^2}} < \frac{\pi}{2}$$

(e) Derive the condition on $V_0$ for there to be one solution (bound state) for $-V_0 < E < 0$.

one solution: $\frac{\pi}{2} < \sqrt{\frac{2mV_0(b-a)^2}{\hbar^2}} < \frac{3\pi}{2}$