Name:

Final Exam - PHY 4604 - Fall 2011
December 16, 2011
10:00A-12:00P, NPB 1002

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

Each exam question, (a),..., (d), is worth 5 points, and the entire exam is out of 100 points. Some formulas are given with the relevant question. The parts marked ALC question are multiple choice, but if you show your work in addition to circling your answer you may receive partial credit.

1. **Short answer:**

(a) Write down the time dependent Schrodinger equation in one dimension.

$$i\hbar \frac{\partial \psi}{\partial t} = \frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + V(x)\psi$$

(b) Including spin, what is the total number of states with $n = 2$ for the Hydrogen atom?

For $n = 2$ there are $\underbrace{l=0}_{2\times 1}$ and $\underbrace{l=1}_{2\cdot 3}$ states.

$2\times 1 + 2\cdot 3 = 8$ states

(c) Write down the wave function for two identical Fermions where one particle is in the state $\psi_a(r)$ and the other is in state $\psi_b(r)$.

$$\psi(r_1, r_2) = \frac{1}{\sqrt{2}}\left(\psi_a(r_1)\psi_b(r_2) - \psi_a(r_2)\psi_b(r_1)\right)$$

(d) Let $\psi_n(x)$ for $n = 1, 2, \ldots$ be a complete orthonormal set of states. At some time the system is in state $\phi(x)$, where

$$\langle\phi|\psi_1\rangle = \frac{1}{\sqrt{2}}$$
$$\langle\phi|\psi_2\rangle = \frac{1}{\sqrt{3}}$$

At this time what is the probability of the system being in a state other than ψ_1 or ψ_2?

$$|\langle\phi|\psi_1\rangle|^2 + |\langle\phi|\psi_2\rangle|^2 = \frac{1}{2} + \frac{1}{3} = \frac{5}{6}$$

Prob. not in 1 or 2 is $\frac{1}{6}$.

2. **One Dimensional Schrodinger Equation:**
Consider the potential

$$\begin{aligned} V(x) &= 0 \text{ for } x < -a \\ V(x) &= V_0 \text{ for } -a < x < a \\ V(x) &= V_1 \text{ for } a < x, \end{aligned}$$

where $V_0 > V_1 > 0$. For an energy E satisfying $V_0 > E > V_1$ the general form of the solution to the time independent Schrodinger equation is

$$\psi(x < -a) = Ae^{ikx} + A'e^{-ikx} \quad (1)$$
$$\psi(a < x < -a) = Be^{\rho x} + B'e^{-\rho x} \quad (2)$$
$$\psi(a < x) = Ce^{ik'x} + C'e^{-ik'x} \quad (3)$$

Assume the particle has mass m.

(a) What are k, ρ, k' in terms of E? For wave vector k what is the momentum, energy, and wavelength of the particle?

$$k = \sqrt{\frac{2mE}{\hbar^2}} \qquad \rho = \sqrt{\frac{2m(V_0 - E)}{\hbar^2}} \qquad k' = \sqrt{\frac{2m(E - V_1)}{\hbar^2}}$$

$$k = \frac{2\pi}{\lambda} \rightarrow \lambda = 2\pi/k$$

$$p = \hbar k$$

$$E = \frac{\hbar^2 k^2}{2m}$$

(b) (ALC question) Consider the following three particles:
1. a free electron with kinetic energy E_o
2. a free proton with kinetic energy E_o
3. a free proton with kinetic energy $2E_o$.

Rank them according to the wavelengths of their matter wave, least to greatest.

A. 1, 2, 3
B. 3, 2, 1
C. 2, 3, 1
D. 1, 3, 2
E. 1, 2, 3

$$\lambda = \frac{2\pi}{k} = 2\pi\sqrt{\frac{\hbar^2}{2mE}}$$

$m_e \ll m_p \rightarrow \lambda_e \gg \lambda_p$ for same E

$\lambda \propto 1/\sqrt{E}$

(c) In terms of $A, \ldots, C'$ what are the boundary conditions at $x = -a$ and $x = a$?

$$\psi(a) = C e^{ik'a} + C' e^{-ik'a} = B e^{\rho a} + B' e^{-\rho a}$$

$$\psi'(a) = ik'(C e^{ik'a} - C' e^{-ik'a}) = \rho (B e^{\rho a} - B' e^{-\rho a})$$

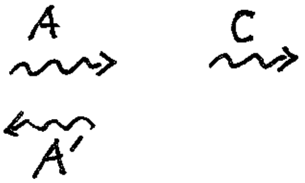
$$\psi(-a) = A e^{-ika} + A' e^{ika} = B e^{-\rho a} + B' e^{+\rho a}$$

$$\psi'(-a) = ik(A e^{-ika} - A' e^{ika}) = \rho (B e^{-\rho a} - B' e^{\rho a})$$

(d) Suppose after solving the boundary conditions in part (c) you found the relation

$$\begin{pmatrix} A \\ A' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} C \\ C' \end{pmatrix}, \tag{4}$$

where α, β, γ, δ are complex numbers. What are the transmission and reflection probabilities for a wave going from left to right in terms of α, β, γ, δ?



$$C' = 0 \Rightarrow \begin{aligned} A &= \alpha C \\ A' &= \gamma C \end{aligned}$$

$$R = \left|\frac{A'}{A}\right|^2 = \frac{|\gamma|^2}{|\alpha|^2}$$

$$T = \frac{k'}{k}\left|\frac{C}{A}\right|^2 = \frac{k'}{k}\frac{1}{|\alpha|^2} = 1 - R$$

3. **Harmonic Oscillator:**
The Hamiltonian of the one-dimensional harmonic oscillator is

$$H = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{1}{2}m\omega^2 x^2 = \frac{p_x^2}{2m} + \frac{1}{2}m\omega^2 x^2 \tag{5}$$

(a) What are the energy eigenvalues of this Hamiltonian?

$$E_n = \hbar\omega\left(n + \frac{1}{2}\right) \quad \text{for } n = 0, 1, 2, \ldots$$

(b) Suppose state of system at some time is

$$\psi(x) = \frac{1}{\sqrt{2}}(\psi_2(x) + i\psi_3(x)), \tag{6}$$

where $\psi_n(x)$ for $n = 0, 1, 2, \ldots$ are the energy eigenstates of the Hamiltonian. Compute the expectation values of x and p, where

$$\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-) \\ p &= i\sqrt{\frac{\hbar m\omega}{2}}(a_+ - a_-). \end{aligned}$$

Using the Dirac notation

$$\langle x \rangle = \langle \psi | x | \psi \rangle = \frac{1}{\sqrt{2}}\left(\langle \psi_2 | - i \langle \psi_3 |\right)\sqrt{\frac{\hbar}{2m\omega}}(a_+ + a_-)\frac{1}{\sqrt{2}}\left(|\psi_2\rangle + i|\psi_3\rangle\right)$$

$$= \frac{1}{2}\sqrt{\frac{\hbar}{2m\omega}}\left(i\underbrace{\langle \psi_2 | a_- | \psi_3 \rangle}_{\sqrt{3}} - i\underbrace{\langle \psi_3 | a_+ | \psi_2 \rangle}_{\sqrt{3}}\right) = 0$$

$$\langle p \rangle = \langle \psi | p | \psi \rangle = \frac{1}{2}\sqrt{\frac{\hbar m\omega}{2}}\left(i(-i)\langle \psi_2 | a_- | \psi_3 \rangle - i(i)\langle \psi_3 | a_+ | \psi_2 \rangle\right)$$

$$= \sqrt{3}\sqrt{\frac{\hbar m\omega}{2}}$$

(c) Compute the expectation values of x^2 and p^2. Verify that the uncertainty principle is satisfied.

$$x^2 = \frac{\hbar}{2m\omega}\left(a_+^2 + a_-^2 + a_+a_- + a_-a_+\right)$$

$$p^2 = \frac{\hbar m\omega}{2}\left(a_+a_- + a_-a_+ - a_+^2 - a_-^2\right)$$

$$a_+a_-|\psi_n\rangle = n|\psi_n\rangle \qquad a_-a_+|\psi_n\rangle = (n+1)|\psi_n\rangle$$

For this ψ the a_+^2 & a_-^2 terms do not contribute.

$$\langle\psi|x^2|\psi\rangle = \frac{\hbar^2}{2m\omega}\frac{1}{\sqrt{2}}\left(\langle\psi_2| - i\langle\psi_3|\right)\frac{1}{\sqrt{2}}\left(5|\psi_2\rangle + i7|\psi_3\rangle\right) = 6\frac{\hbar}{2m\omega}$$

$$\langle\psi|p^2|\psi\rangle = 6\frac{\hbar m\omega}{2}$$

$$\Rightarrow \Delta x = \sqrt{3}\sqrt{\frac{\hbar}{m\omega}}, \quad \Delta p = \sqrt{3-\frac{3}{2}}\sqrt{\hbar m\omega} = \sqrt{\frac{3}{2}}\sqrt{\hbar m\omega}$$

$$\Rightarrow \Delta x \Delta p = \frac{3}{\sqrt{2}}\hbar > \frac{\hbar}{2} \checkmark$$

(d) (ALC question) The Hamiltonian of a particle of mass m in a two dimensional isotropic harmonic potential is represented by

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} + \frac{1}{2}m\omega^2(x^2+y^2). \tag{7}$$

The energy level of the particle is (n_x and n_y are non-negative integers):

A. $\hbar\omega(n_x + n_y)$
B. $\hbar\omega(\frac{1}{2} + n_x + n_y)$
C. $\hbar\omega(n_x + n_y)$
D. $\hbar\omega(1 + n_x + n_y)$ $= \hbar\omega(n_x + \frac{1}{2}) + \hbar\omega(n_y + \frac{1}{2})$
E. $2\hbar\omega(n_x + n_y)$

4. **Formalism:**
Consider the Hamiltonian for a two state system represented by the matrix

$$H = E_0 \begin{pmatrix} 1 & e^{i\varphi} \\ e^{-i\varphi} & -1 \end{pmatrix}. \tag{8}$$

(a) What are the energy eigenvalues and eigenvectors of H?

$$\det\begin{pmatrix} 1-\lambda & e^{i\varphi} \\ e^{-i\varphi} & -1-\lambda \end{pmatrix} = 0 \to \lambda^2 - 1 - 1 = 0 \to \lambda = \pm\sqrt{2}$$

$$\lambda = +\sqrt{2} \to E = \sqrt{2}E_0, \quad \begin{pmatrix} 1-\sqrt{2} & e^{i\varphi} \\ e^{-i\varphi} & -1-\sqrt{2} \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0$$

$$\to \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = C\begin{pmatrix} 1+\sqrt{2} \\ e^{-i\varphi} \end{pmatrix}.$$

$$\lambda = -\sqrt{2} \to E = -\sqrt{2}E_0, \quad \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = C\begin{pmatrix} -e^{i\varphi} \\ 1+\sqrt{2} \end{pmatrix},$$

where $C = \{1 + (1+\sqrt{2})^2\}^{-1/2}$

(b) Calculate the projection operators for the eigenvectors in part (a) and verify that they satisfy the completeness relation.

$$P_{\sqrt{2}E_0} = C^2\begin{pmatrix} 1+\sqrt{2} \\ e^{-i\varphi} \end{pmatrix}\begin{pmatrix} 1+\sqrt{2} & e^{i\varphi} \end{pmatrix} = C^2\begin{pmatrix} (1+\sqrt{2})^2 & (1+\sqrt{2})e^{i\varphi} \\ (1+\sqrt{2})e^{-i\varphi} & 1 \end{pmatrix}$$

$$P_{-\sqrt{2}E_0} = C^2\begin{pmatrix} -e^{+i\varphi} \\ 1+\sqrt{2} \end{pmatrix}\begin{pmatrix} -e^{-i\varphi} & 1+\sqrt{2} \end{pmatrix} = C^2\begin{pmatrix} 1 & (1+\sqrt{2})e^{-i\varphi} \\ -(1+\sqrt{2})e^{-i\varphi} & (1+\sqrt{2})^2 \end{pmatrix}$$

$$P_{\sqrt{2}E_0} + P_{-\sqrt{2}E_0} = C^2\begin{pmatrix} 1+(1+\sqrt{2})^2 & 0 \\ 0 & 1+(1+\sqrt{2})^2 \end{pmatrix} = \mathbb{1}$$

(c) Suppose at time $t = 0$ the system is in the state

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{9}$$

What is the state of the system at time t?

$$C\,(1+\sqrt{2} \quad e^{i\varphi})\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C e^{i\varphi}$$

$$C\,(-e^{-i\varphi} \quad 1+\sqrt{2})\begin{pmatrix} 0 \\ 1 \end{pmatrix} = C(1+\sqrt{2})$$

$$\to \psi(t) = C^2 e^{i\varphi} e^{-i\sqrt{2}E_0 t/\hbar}\begin{pmatrix} 1+\sqrt{2} \\ e^{-i\varphi} \end{pmatrix} + C^2(1+\sqrt{2})\, e^{i\sqrt{2}E_0 t/\hbar}\begin{pmatrix} -e^{i\varphi} \\ 1+\sqrt{2} \end{pmatrix}$$

$$= C^2 \begin{pmatrix} (1+\sqrt{2})e^{i\varphi}\left(e^{-i\sqrt{2}E_0 t/\hbar} - e^{i\sqrt{2}E_0 t/\hbar}\right) \\ e^{-i\sqrt{2}E_0 t/\hbar} + (1+\sqrt{2})e^{i\sqrt{2}E_0 t/\hbar} \end{pmatrix}$$

(d) What is the probability of being in the state

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} \tag{10}$$

at time t?

$$\left|C^2 (1+\sqrt{2})e^{i\varphi}\left(e^{-i\sqrt{2}E_0 t/\hbar} - e^{i\sqrt{2}E_0 t/\hbar}\right)\right|^2 =$$

$$= C^4 (1+\sqrt{2})^2 \left(1 + 1 - 2\cos(2\sqrt{2}E_0 t/\hbar)\right)$$

$$= \underline{\frac{2(1+\sqrt{2})^2 (1-\cos(2\sqrt{2}E_0 t/\hbar))}{(1+(1+\sqrt{2})^2)^2}}$$

5. **Angular momentum:**

(a) For a spin 1/2 particle what is the relation between the spin, $\vec{S}$, and the Pauli spin matrices, $\vec{\sigma}$?

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_y = \begin{pmatrix} 0 & -i \\ +i & 0 \end{pmatrix}, \quad \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

$$\vec{S} = \frac{\hbar}{2} \vec{\sigma}$$

(b) Compute the commutator $[L_x, L_x L_y L_x]$?

$$L_x L_x L_y L_x - L_x L_y L_x L_x$$

$$= L_x [L_x, L_y] L_x$$

$$= i\hbar L_x L_z L_x$$

(c) For $j = 3/2$ and $m = 1/2$ what is the expectation value $\langle j, m | J_x J_y | j, m \rangle$?

$$J_{\pm} = J_x \pm i J_y \rightarrow J_x = (J_+ + J_-)/2$$
$$J_y = (J_+ - J_-)/2i$$
$$\rightarrow J_x J_y = \frac{1}{4i}\left(J_+^2 - J_-^2 + J_- J_+ - J_+ J_-\right)$$

$$J_+ J_- |j, m\rangle = \hbar \sqrt{j(j+1) - m(m-1)}\, J_+ |j, m-1\rangle$$
$$= \hbar^2 \sqrt{j(j+1) - m(m-1)} \sqrt{j(j+1) - (m-1)m}\, |j, m\rangle$$
$$= \hbar^2 \left(j(j+1) - m(m-1)\right) |j, m\rangle$$

(d) From the table of Clebsh-Gordon coefficients below what is the $j = 2$, $m = 2$ state created from adding $j_1 = 2$ and $j_2 = 1$ angular momentum states?

2×1		3
		+3
+2	+1	1

		3	2
		+2	+2
+2	0	1/3	2/3
+1	+1	2/3	−1/3

$$|2,2\rangle = \sqrt{\frac{2}{3}}\, |2,2\rangle \otimes |1,0\rangle - \sqrt{\frac{1}{3}}\, |2,1\rangle \otimes |1,1\rangle$$

$$J_- J_+ |j, m\rangle = \hbar \sqrt{j(j+1) - m(m+1)}\, J_- |j, m+1\rangle$$
$$= \hbar^2 \sqrt{j(j+1) - m(m+1)} \sqrt{j(j+1) - (m+1)m}\, |j, m\rangle$$
$$= \hbar^2 \left(j(j+1) - m(m+1)\right) |j, m\rangle$$

$$\Rightarrow (J_- J_+ - J_+ J_-) |j, m\rangle = \hbar^2 \left(m(m-1) - m(m+1)\right) |j, m\rangle = -2m\hbar^2 |j, m\rangle$$

$$\Rightarrow \langle \tfrac{3}{2}, \tfrac{1}{2} | J_x J_y | \tfrac{3}{2}, \tfrac{1}{2} \rangle = \frac{-\hbar^2}{4i}$$