Free Particle

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} = E\psi$$

$$\frac{d^2\psi}{dx^2} = -k^2\psi \quad , \; k = \frac{\sqrt{2mE}}{\hbar} \quad , \; E_k = \frac{\hbar^2 k^2}{2m}$$

Solutions: $\psi(x) = A e^{ikx} + B e^{-ikx}$

Note: not normalized (or normalizable).

$$\Psi(x,t) = A e^{ikx} e^{-i\frac{E_k}{\hbar}t} + B e^{-ikx} e^{-i\frac{E_k}{\hbar}t}$$

$$= A e^{ik(x - \frac{\hbar k}{2m}t)\cancel{t}} + B e^{-ik(x + \frac{\hbar k}{2m}t)}$$

Looks like $v = \frac{\hbar k}{2m}$ not $\frac{\hbar k}{m}$. Come back to this later today.

To normalize make a wave packet

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k)\, e^{i(kx - \frac{\hbar k^2}{2m}t)}\, dk$$

Griffiths' convention

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(x)\, e^{-ikx}\, dx \quad \text{... Fourier transform}$$

$$\Updownarrow$$

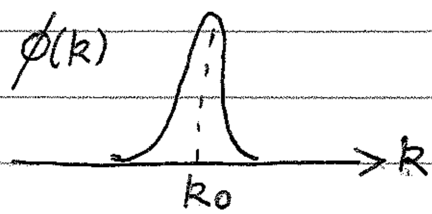
$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} F(k)\, e^{ikx}\, dk \quad \text{... inverse Fourier transform}$$

Apply:

$$\psi(x,0) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k)\, e^{ikx}\, dk$$

$$\Rightarrow \phi(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \psi(x,0)\, e^{-ikx}\, dx$$

Suppose $\phi(k)$ is peaked about some k_0



Expand ω about k_0:

$$\omega(k) \approx \omega_0 + \underbrace{\left.\frac{d\omega}{dk}\right|_{k_0}}_{\equiv \omega_0'} (k - k_0)$$

$$\Psi(x,t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k) e^{i(kx-\omega t)} \, dk$$

$$\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k_0+s)\, e^{i((k_0+s)x - (\omega_0 + \omega_0' s)t)} \, ds$$

$$e^{i\omega_0' k_0 t} \qquad e^{-i\omega_0' k_0 t}$$

$$\Psi(x,0) \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} \phi(k_0+s)\, e^{i(k_0+s)x} \, ds$$

$$\Psi(x,t) \approx \frac{1}{\sqrt{2\pi}} e^{i(-\omega_0 t + k_0\omega_0' t)} \int_{-\infty}^{+\infty} \phi(k_0+s)\, e^{i(k_0+s)(x-\omega_0' t)} \, ds$$

$$\approx e^{-i(\omega_0 - k_0\omega_0')t}\, \Psi(x-\omega_0' t, 0)$$

velocity of packet $= \left.\frac{d\omega}{dk}\right|_{k_0} \equiv v_{group} = \frac{\hbar k}{m}$ ✓

not $\frac{\omega}{k} = v_{phase}$.

Note that for E&M waves

$v_{phase} = \frac{\omega}{k} = c = v_{group} = \frac{d\omega}{dk}$

because the dispersion is linear.