Review last time:

$$H\psi = -\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \frac{1}{2}m\omega^2 x^2\psi$$

$$= \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)\psi,$$

where $a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x)$

$$a_- = \frac{1}{\sqrt{2\hbar m\omega}}(ip + m\omega x).$$

Commutation relations: $[x, p] = i\hbar$
$[a_-, a_+] = 1$

Raising & lowering operators:

If $H\psi = E\psi$, then $a_+\psi$ is also an eigenstate w/ energy $E + \hbar\omega$, and $a_-\psi$ has energy $E - \hbar\omega$.

Energy bounded below $\rightarrow$ There is ψ_0 such that

$$a_-\psi_0 = 0.$$

ground state

Solve for ψ_0:

$$a_- \psi_0 = \frac{1}{\sqrt{2\hbar m\omega}}\left(-i\frac{\hbar}{i}\frac{d}{dx} + m\omega x\right)\psi_0 = 0$$

$$\rightarrow \frac{d\psi_0}{dx} = \frac{-m\omega}{\hbar} x\, \psi_0$$

This is a first order differential equation with solution

$$\psi_0(x) = A e^{-\frac{m\omega}{2\hbar}x^2}.$$

"A" is determined by the normalization condition:

$$1 = \int_{-\infty}^{+\infty} |\psi_0(x)|^2 dx = A^2 \int_{-\infty}^{+\infty} e^{-\frac{m\omega}{\hbar}x^2} dx$$

$$= A^2\left(\frac{\hbar}{m\omega}\right)^{1/2} \underbrace{\int_{-\infty}^{+\infty} e^{-u^2} du}_{\sqrt{\pi}} = A^2\left(\frac{\hbar\pi}{m\omega}\right)^{1/2}$$

$$\rightarrow A = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \text{and } \boxed{\psi_0(x) = \left(\frac{m\omega}{\hbar\pi}\right)^{1/4} \exp\left(-\frac{m\omega}{2\hbar}x^2\right)}$$

Because $a_- \psi_0 = 0$,

$$H\psi_0 = \hbar\omega\left(a_+ a_- + \frac{1}{2}\right)\psi_0 = \frac{\hbar\omega}{2}\psi_0 \rightarrow \boxed{E_0 = \frac{\hbar\omega}{2}}$$

General solution:

$$\boxed{\begin{aligned} \psi_n(x) &= A_n (a_+)^n \psi_0(x) \\ E_n &= \left(n+\tfrac{1}{2}\right)\hbar\omega, \quad n=0,1,2,\ldots \end{aligned}}$$

$$\boxed{a_+ a_- \psi_n = n\psi_n}$$

Determine the normalization constants A_n:

$$\int_{-\infty}^{+\infty} f^*(x)\, a_+ g(x)\, dx = \frac{1}{\sqrt{2\hbar m\omega}} \int_{-\infty}^{+\infty} f^* \left(-\hbar\frac{d}{dx} + m\omega x\right) g\, dx$$

$$= \frac{1}{\sqrt{2\hbar m\omega}} \int_{-\infty}^{+\infty} f^* \left(-\hbar\frac{dg}{dx}\right) dx + \frac{1}{\sqrt{2m\hbar\omega}} \int_{-\infty}^{+\infty} f^* m\omega x\, g\, dx$$

$$= \frac{1}{\sqrt{2\hbar m\omega}} \int_{-\infty}^{+\infty} \hbar\left(\frac{df^*}{dx}\right) g\, dx + \frac{1}{\sqrt{2m\hbar\omega}} \int_{-\infty}^{+\infty} m\omega x f^* g\, dx$$

integration by parts

$$= \frac{1}{\sqrt{2\hbar m\omega}} \int_{-\infty}^{+\infty} \left(\left(\hbar\frac{d}{dx} + m\omega x\right) f^*\right) g\, dx$$

$$= \int (a_- f)^* g\, dx$$

$$\Rightarrow \boxed{\int f^* (a_+ g)\, dx = \int (a_- f)^* g\, dx}$$

Similarly, $\boxed{\int f^* (a_- g)\, dx = \int (a_+ f)^* g\, dx}$

$$\int (a_+ \psi_n)^* (a_+ \psi_n)\, dx = \int \psi_n^* (a_- a_+ \psi_n)\, dx$$

$$= (n+1) \int \psi_n^* \psi_n \, dx = n+1$$

because $[a_-, a_+] = 1 \rightarrow a_- a_+ = a_+ a_- + 1$.

Since $\psi_{n+1} \propto a_+ \psi_n$ or $c_n \psi_{n+1} = a_+ \psi_n$,
we see that $c_n^2 = n+1$ or $c_n = \sqrt{n+1}$.

$$\boxed{a_+ \psi_n = \sqrt{n+1}\ \psi_{n+1}}$$

$$\boxed{a_- \psi_n = \sqrt{n}\ \psi_{n-1}}$$

Apply repeatedly,

$$a_+ \psi_0 = \sqrt{1}\ \psi_1 \longrightarrow \psi_1 = \frac{1}{\sqrt{1}} a_+ \psi_0$$
$$a_+ \psi_1 = \sqrt{2}\ \psi_2 \longrightarrow \psi_2 = \frac{1}{\sqrt{2}} a_+ \psi_1 = \frac{1}{\sqrt{2!}} a_+^2 \psi_0$$
$$a_+ \psi_2 = \sqrt{3}\ \psi_3 \longrightarrow \psi_3 = \frac{1}{\sqrt{3}} a_+ \psi_2 = \frac{1}{\sqrt{3!}} a_+^3 \psi_0$$
$$\vdots$$

$$\boxed{\psi_n = \frac{1}{\sqrt{n!}} (a_+)^n \psi_0}$$

The ψ_n are orthogonal:

For $m \neq n$, $\int_{-\infty}^{+\infty} \psi_m^* (a_+ a_- \psi_n) dx = n \int_{-\infty}^{+\infty} \psi_m^* \psi_n dx$

$$= \int_{-\infty}^{+\infty} (a_- \psi_m)^* (a_- \psi_n) dx$$

$$= \int_{-\infty}^{+\infty} (a_+ a_- \psi_m)^* \psi_n dx = m \int_{-\infty}^{+\infty} \psi_m^* \psi_n dx$$

$$\rightarrow \quad 0 = \int \psi_m^* \psi_n dx.$$

Expectation values:

Using
$$a_+ = \frac{1}{\sqrt{2\hbar m\omega}}(-ip + m\omega x)$$
$$a_- = \frac{1}{\sqrt{2\hbar m\omega}}(ip + m\omega x),$$

we can derive

$$\boxed{\begin{aligned} x &= \sqrt{\frac{\hbar}{2m\omega}}\,(a_+ + a_-) \\ p &= i\sqrt{\frac{\hbar m\omega}{2}}\,(a_+ - a_-) \end{aligned}}$$

and from these we can compute expectation values of x & p, x^2, p^2, etc.