Name:

Quiz 4

Due Monday at 9:35am.

Consider the time independent Schrodinger equation with two delta function potentials:

$$\frac{-\hbar^2}{2m}\frac{d^2\psi}{dx^2} + \alpha\delta(x-a)\psi(x) + \alpha\delta(x+a)\psi(x) = E\psi(x), \tag{1}$$

where both E and α are negative. In solving this problem you may attach extra paper or work on the back of this sheet. Please clearly box your answers.

1. What is the form of the solution to the Schrodinger equation in each of the three regions: $x < -a$, $-a < x < a$, and $x > a$? Assume that the wavefunction is normalizable.

Let $\lambda = \sqrt{2m(-E)/\hbar^2}$

$$\psi(x>a) = Ae^{-\lambda x}$$
$$\psi(-a<x<a) = Be^{\lambda x} + Ce^{-\lambda x}$$
$$\psi(x<-a) = De^{\lambda x}$$

2. Because the potential is an even function, we can assume that the solutions are either even or odd functions of x. Assume ψ is an even function of x: $\psi(x) = \psi(-x)$. What is the general form of an even solution for $\psi(x)$?

$$\psi(x>a) = Ae^{-\lambda x}$$
$$\psi(x<-a) = Ae^{\lambda x}$$
$$\psi(-a<x<a) = B\cosh(\lambda x)$$

3. What are the boundary conditions for the wave function in part (b)?

The boundary conditions at $x = +a$ & $-a$ give the same equations.

$$\psi(a) = Ae^{-\lambda a} = B\cosh(\lambda a)$$
$$\psi'(a^+) - \psi'(a^-) = -\lambda Ae^{-\lambda a} - \lambda B\sinh(\lambda a) = \frac{2m\alpha}{\hbar^2}\psi(a)$$

4. Use the boundary conditions to derive an equation to find the even bound states. Solve this equation graphically. How many solutions are there?

$\Longrightarrow$

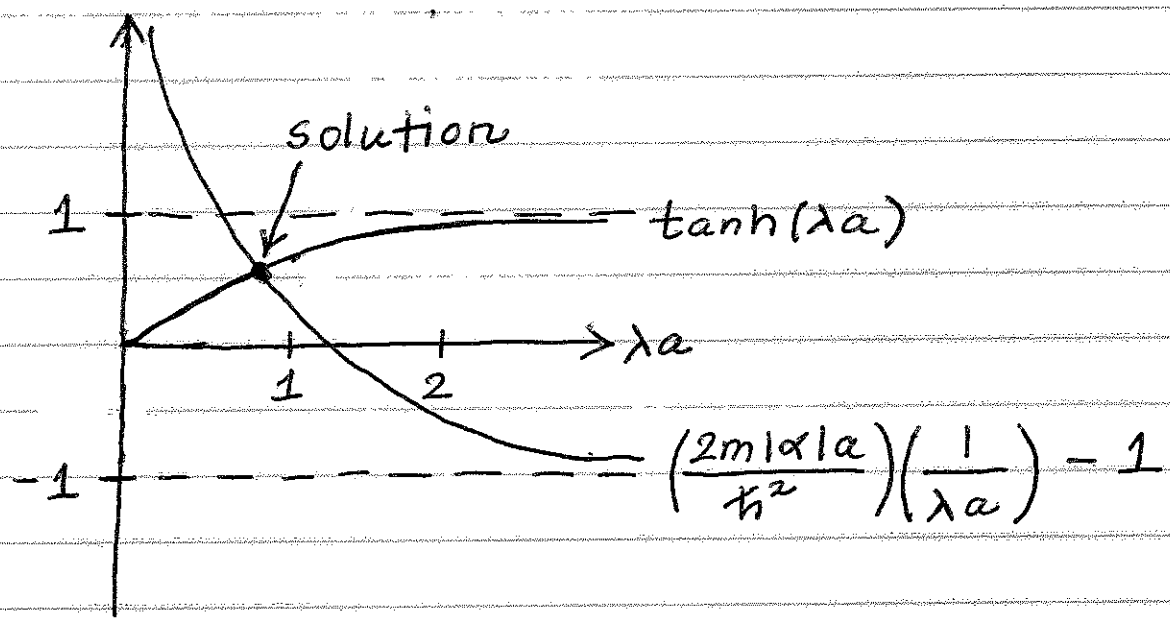
4. Substitute the 1^{st} boundary condition into the second:

$$-\lambda \cancel{B}\cosh(\lambda a) - \lambda \cancel{B}\sinh(\lambda a) = \frac{2m\alpha}{\hbar^2}\cancel{B}\cosh(\lambda a)$$

$$\rightarrow \left(\frac{-2m\alpha}{\hbar^2} - \lambda\right)\cosh(\lambda a) = \lambda \sinh(\lambda a)$$

$$\rightarrow \tanh(\lambda a) = \frac{-2m\alpha}{\hbar^2 \lambda} - 1$$

$$\text{or } \boxed{\tanh(\lambda a) = \left(\frac{2m|\alpha|a}{\hbar^2}\right)\left(\frac{1}{\lambda a}\right) - 1}$$



There is one even solution for each $\alpha < 0$.

Many of you derived an equivalent relation:

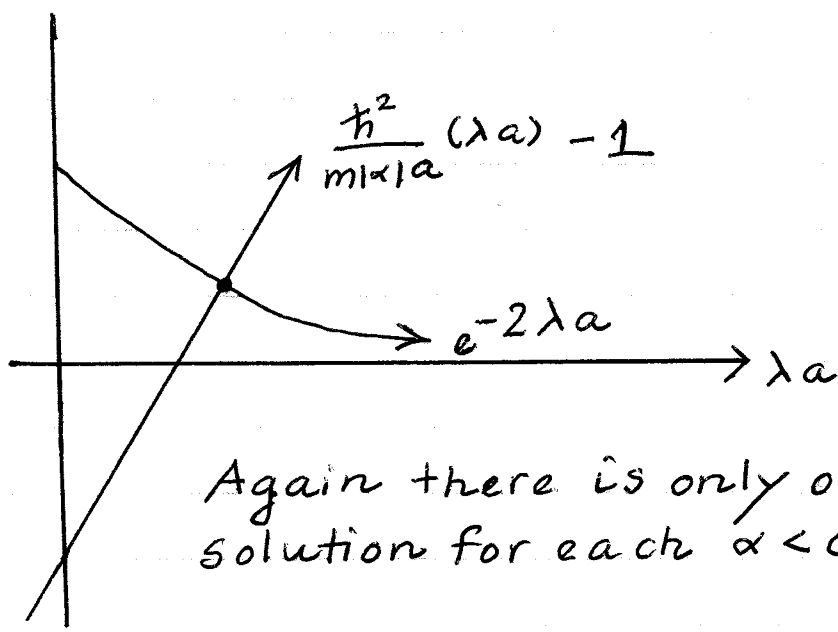
$$-\lambda \cancel{B} \cosh(\lambda a) - \cancel{\lambda B} \sinh(\lambda a) = \frac{2m\alpha}{\hbar^2} \cancel{B} \cosh(\lambda a)$$

$$\rightarrow \quad -\lambda e^{\lambda a} = \frac{\cancel{2}m\alpha}{\hbar^2} \left(\frac{e^{\lambda a} + e^{-\lambda a}}{\cancel{2}} \right)$$

$$\rightarrow \quad -\lambda = \frac{m\alpha}{\hbar^2} \left(1 + e^{-2\lambda a} \right)$$

$$\frac{\hbar^2}{m|\alpha|} \lambda - 1 = e^{-2\lambda a}$$

$$\boxed{\left(\frac{\hbar^2}{m|\alpha| a} \right) (\lambda a) - 1 = e^{-2(\lambda a)}}$$



Again there is only one even solution for each $\alpha < 0$.