There are four versions of this quiz. 1.

Name:

Quiz 6

Consider the following three Hermitian operators:

$$L_x = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \; L_y = \frac{\hbar}{\sqrt{2}}\begin{pmatrix} 0 & -i & 0 \\ i & 0 & -i \\ 0 & i & 0 \end{pmatrix} \text{ and } L_z = \hbar\begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{1}$$

(We will see that these correspond to the angular momentum operators for $l = 1$, but for now just treat them as Hermitian operators.)

or L_y

1. If the state of the system just prior to an L_x measurement is

$$|\psi(0^-)\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \tag{2}$$

what are the possible outcomes of the L_x measurement and what are the probabilities of those outcomes?

2. If many identical measurements were made, what would be the average value of L_x obtained?

or $-\hbar$

3. Suppose the measurement in part 1. yields a value of $+\hbar$. What is the state of the system immediately after the measurement?

4. If the Hamiltonian of the system is $H = \omega L_z$, what is the state of the system at time t later?

or L_x

5. A measurement of L_y is made at time t, what are the possible outcomes and the probabilities of those outcomes?

Quiz 6 Solution

1.

L_x measurement

outcome	probability
0	1/2
$+\hbar$	1/4
$-\hbar$	1/4

L_y measurement

outcome	probability
0	1/2
$+\hbar$	1/4
$-\hbar$	1/4

Probability $= |\langle \text{eigenvec. of outcome} | \psi \rangle|^2$

2. $\langle \psi(0^-)|L_x|\psi(0^-)\rangle = 0$ $\qquad$ $\langle \psi(0^-)|L_y|\psi(0^-)\rangle = 0$

3.

$\hbar$	$-\hbar$	$\hbar$	$-\hbar$
$\lvert\psi(0^+)\rangle = \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} -i \\ \sqrt{2} \\ i \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} i \\ \sqrt{2} \\ -i \end{pmatrix}$

4.

$\lvert\psi(t)\rangle = \frac{1}{2}\begin{pmatrix} e^{-i\omega t} \\ \sqrt{2} \\ e^{i\omega t} \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} e^{-i\omega t} \\ -\sqrt{2} \\ e^{i\omega t} \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} -ie^{-i\omega t} \\ \sqrt{2} \\ ie^{i\omega t} \end{pmatrix}$	$\frac{1}{2}\begin{pmatrix} ie^{-i\omega t} \\ \sqrt{2} \\ -ie^{i\omega t} \end{pmatrix}$

5.

	L_y meas.	L_y meas.	L_x meas.	L_x meas.
outcome	prob.	prob.	prob.	prob.
0	$\frac{1}{2}\cos^2(\omega t)$	$\frac{1}{2}\cos^2(\omega t)$	$\frac{1}{2}\cos^2(\omega t)$	$\frac{1}{2}\cos^2(\omega t)$
$+\hbar$	☺ $\frac{1}{4}(1+\sin(\omega t))^2$	$\frac{1}{4}(1-\sin(\omega t))^2$	☺ $\frac{1}{4}(1-\sin(\omega t))^2$	$\frac{1}{4}(1+\sin(\omega t))^2$
$-\hbar$	$\frac{1}{4}(1-\sin(\omega t))^2$	$\frac{1}{4}(1+\sin(\omega t))^2$	$\frac{1}{4}(1+\sin(\omega t))^2$	$\frac{1}{4}(1-\sin(\omega t))^2$

Eigenvectors & Eigenvalues of L_x:

$$\det\begin{pmatrix} -\lambda & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & -\lambda & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & -\lambda \end{pmatrix} = -\lambda^3 + \frac{\hbar^2}{2}\lambda + \frac{\hbar^2}{2}\lambda = 0$$

$$= -\lambda^3 + \hbar^2\lambda$$

$$= -(\lambda^2 - \hbar^2)\lambda$$

$$= -(\lambda-\hbar)(\lambda+\hbar)\lambda$$

$$\rightarrow \lambda = 0, \pm\hbar$$

eigenvector

$\underline{\lambda = 0:}$ $$\begin{pmatrix} 0 & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & 0 & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0 \rightarrow \begin{matrix} C_2 = 0 \\ C_1 + C_3 = 0 \end{matrix} \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$$

$\underline{\lambda = \hbar:}$ $$\begin{pmatrix} -\hbar & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & -\hbar & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & -\hbar \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0 \rightarrow \begin{matrix} C_2 = \sqrt{2}\,C_1 \\ = \sqrt{2}\,C_3 \end{matrix} \rightarrow \frac{1}{2}\begin{pmatrix} 1 \\ \sqrt{2} \\ 1 \end{pmatrix}$$

$\underline{\lambda = -\hbar:}$ $$\begin{pmatrix} \hbar & \hbar/\sqrt{2} & 0 \\ \hbar/\sqrt{2} & \hbar & \hbar/\sqrt{2} \\ 0 & \hbar/\sqrt{2} & \hbar \end{pmatrix}\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix} = 0 \rightarrow \begin{matrix} C_2 = -\sqrt{2}\,C_1 \\ = -\sqrt{2}\,C_3 \end{matrix} \rightarrow \frac{1}{2}\begin{pmatrix} 1 \\ -\sqrt{2} \\ 1 \end{pmatrix}$$

Eigenvectors & Eigenvalues of L_y:

$$\det\begin{pmatrix} -\lambda & -i\hbar/\sqrt{2} & 0 \\ i\hbar/\sqrt{2} & -\lambda & -i\hbar/\sqrt{2} \\ 0 & i\hbar/\sqrt{2} & -\lambda \end{pmatrix} = -\lambda^3 + \frac{\hbar^2}{2}\lambda + \frac{\hbar^2}{2}\lambda = 0$$

$$= -\lambda^3 + \hbar^2\lambda$$

$$\rightarrow \lambda = 0, \pm\hbar$$

eigenvector

$\lambda = 0$:
$$\begin{pmatrix} 0 & -i\hbar/\sqrt{2} & 0 \\ i\hbar/\sqrt{2} & 0 & -i\hbar/\sqrt{2} \\ 0 & i\hbar/\sqrt{2} & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \rightarrow \begin{matrix} c_2 = 0 \\ c_1 - c_3 = 0 \end{matrix} \rightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

$\lambda = \hbar$:
$$\begin{pmatrix} -\hbar & -i\hbar/\sqrt{2} & 0 \\ i\hbar/\sqrt{2} & -\hbar & -i\hbar/\sqrt{2} \\ 0 & i\hbar/\sqrt{2} & -\hbar \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \rightarrow \begin{matrix} c_1 = -i/\sqrt{2}\, c_2 \\ c_2 = i\sqrt{2}\, c_1 \\ c_3 = i/\sqrt{2}\, c_2 \\ c_2 = -i\sqrt{2}\, c_3 \end{matrix} \rightarrow \frac{1}{2}\begin{pmatrix} -i \\ \sqrt{2} \\ i \end{pmatrix}$$

$\lambda = -\hbar$:
$$\begin{pmatrix} \hbar & -i\hbar/\sqrt{2} & 0 \\ i\hbar/\sqrt{2} & \hbar & -i\hbar/\sqrt{2} \\ 0 & i\hbar/\sqrt{2} & \hbar \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = 0 \rightarrow \begin{matrix} c_1 = i/\sqrt{2}\, c_2 \\ c_3 = -i/\sqrt{2}\, c_2 \end{matrix} \rightarrow \frac{1}{2}\begin{pmatrix} i \\ \sqrt{2} \\ -i \end{pmatrix}$$

Calculation of select probabilities:

$$\left|\frac{1}{2}\begin{pmatrix} i & \sqrt{2} & -i \end{pmatrix}\frac{1}{2}\begin{pmatrix} e^{-i\omega t} \\ \sqrt{2} \\ e^{i\omega t} \end{pmatrix}\right|^2 = \left|\frac{1}{4}\left(2 + ie^{-i\omega t} - ie^{i\omega t}\right)\right|^2$$

$$= \left|\frac{1+\sin(\omega t)}{2}\right|^2$$

$$= \frac{1}{4}\left(1+\sin(\omega t)\right)^2$$

$$\left|\frac{1}{2}\begin{pmatrix} 1 & \sqrt{2} & 1 \end{pmatrix}\frac{1}{2}\begin{pmatrix} -ie^{-i\omega t} \\ \sqrt{2} \\ ie^{i\omega t} \end{pmatrix}\right|^2 = \left|\frac{1}{4}\left(2 - ie^{-i\omega t} + ie^{i\omega t}\right)\right|^2$$

$$= \frac{1}{4}\left(1-\sin(\omega t)\right)^2$$