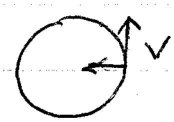
Spin 1/2 Particles:

Review of magnetic moments:

$U = -\vec{M} \cdot \vec{B}$

M = magnetic moment
$= I \cdot A$

(orbital magnetic moment)

Example: 

B out
$F = q\,\vec{v} \times \vec{B}$
$q = -e$

$I = \frac{\text{charge}}{\text{time}} = \frac{e}{T}$ ← period

$\rightarrow \vec{M} = \frac{e}{T}\pi R^2$ into page

$\vec{L} = (mv)R$ out of page
$= m\,\frac{2\pi R^2}{T} = 2m\,\frac{\pi R^2}{T}$

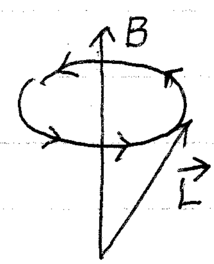
$$\rightarrow \vec{M} = \underbrace{\frac{-e}{2m}}_{} \vec{L}$$

γ = gyromagnetic ratio
For e & m_e, $\gamma\hbar = \mu_B$ = Bohr magneton.

$\vec{F} = -\nabla U = \vec{\nabla}\,\vec{M} \cdot \vec{B}$

Torque $= \vec{M} \times \vec{B} = \frac{d\vec{L}}{dt}$

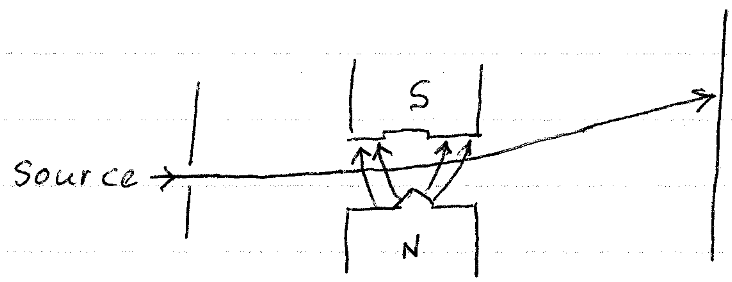
$\rightarrow \frac{d\vec{L}}{dt} = \gamma\,\vec{L} \times \vec{B}$



since $\frac{d\,\vec{L} \cdot \vec{B}}{dt} = 0$

$e \approx 1.6 \times 10^{-19}\,C$
$\hbar \approx 1.05 \times 10^{-34}\,J \cdot s$
$m_e \approx 9.11 \times 10^{-31}\,kg$
$\rightarrow \mu_B = \frac{e\hbar}{2m_e} \approx 9.27 \times 10^{-24}\,A \cdot m^2$

Stern-Gerlach experiment:



$$\vec{F} = \vec{\nabla}(\vec{M}\cdot\vec{B}) \approx \vec{\nabla}(M_z B_z) \approx M_z \vec{\nabla} B_z$$

Classically, $-|M| \leq M_z \leq |M|$.

In experiment see 2 spots for electrons.

Intrinsic angular momentum or spin:

$$S_z|+\rangle = \frac{\hbar}{2}|+\rangle \qquad \vec{M} = g\left(\frac{-e}{2m}\right)\vec{S} \ ; \ g \approx 2$$

$$S_z|-\rangle = -\frac{\hbar}{2}|-\rangle . \qquad U = g\left(\frac{e}{2m}\right)\vec{S}\cdot\vec{B} = g\underbrace{\left(\frac{e\hbar}{2m}\right)}_{\mu_B}\frac{\vec{\sigma}}{2}\cdot\vec{B}$$

In $|+\rangle, |-\rangle$ basis, $S_z = \begin{pmatrix} \langle +|S_z|+\rangle & \langle +|S_z|-\rangle \\ \langle -|S_z|+\rangle & \langle -|S_z|-\rangle \end{pmatrix} = \frac{\hbar}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

$$= \frac{\hbar}{2}\sigma_z$$

Similarly, $S_x = \frac{\hbar}{2}\sigma_x$ & $S_y = \frac{\hbar}{2}\sigma_y$.

General direction:

$$\hat{n} = \sin\theta\cos\varphi\,\hat{x} + \sin\theta\sin\varphi\,\hat{y} + \cos\theta\,\hat{z}$$

$$\hat{n}\cdot\vec{S} = \sin\theta\cos\varphi\, \underset{\frac{\hbar}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}}{S_x} + \sin\theta\sin\varphi\, \underset{\frac{\hbar}{2}\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix}}{S_y} + \cos\theta\, \underset{\frac{\hbar}{2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}}{S_z}$$

$$= \frac{\hbar}{2}\begin{pmatrix}\cos\theta & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta\end{pmatrix}$$

Eigenvectors:

S_x: $\quad |\pm\rangle_x = \frac{1}{\sqrt{2}}\left(|+\rangle \pm |-\rangle\right)$

S_y: $\quad |\pm\rangle_y = \frac{1}{\sqrt{2}}\left(|+\rangle \pm i|-\rangle\right)$

S_z: $\quad |\pm\rangle$

$\hat{n}\cdot\vec{S}$: $\quad \begin{vmatrix}\cos\theta - \lambda & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta - \lambda\end{vmatrix} = 0$

$$\rightarrow\ -(\cos^2\theta - \lambda^2) - \sin^2\theta = 0 \rightarrow \lambda = \pm 1$$

+case: $\begin{pmatrix}\cos\theta - 1 & \sin\theta e^{-i\varphi}\\ \sin\theta e^{i\varphi} & -\cos\theta - 1\end{pmatrix}\begin{pmatrix}c_1\\ c_2\end{pmatrix} = 0 \rightarrow \begin{pmatrix}c_1\\ c_2\end{pmatrix} \propto \begin{pmatrix}1 + \cos\theta\\ \sin\theta e^{i\varphi}\end{pmatrix}.$

$$\cos\theta = \cos^2\left(\frac{\theta}{2}\right) - \sin^2\left(\frac{\theta}{2}\right) = 2 \; - 2\sin^2\left(\frac{\theta}{2}\right)$$

$$= 2\cos^2\left(\frac{\theta}{2}\right) - 1 \rightarrow 2\cos^2\left(\frac{\theta}{2}\right) = \cos\theta + 1$$

$$= 1 - 2\sin^2\left(\frac{\theta}{2}\right) \rightarrow 2\sin^2\left(\frac{\theta}{2}\right) = 1 - \cos\theta$$

$$\sin\theta = 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$\rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \propto \begin{pmatrix} 2\cos^2\left(\frac{\theta}{2}\right) \\ 2\cos\left(\frac{\theta}{2}\right)\sin\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix}$$

$$\propto \begin{pmatrix} \cos\left(\frac{\theta}{2}\right) \\ \sin\left(\frac{\theta}{2}\right)e^{i\varphi} \end{pmatrix}$$

$$\propto \begin{pmatrix} \cos\left(\frac{\theta}{2}\right)e^{-i\varphi/2} \\ \sin\left(\frac{\theta}{2}\right)e^{i\varphi/2} \end{pmatrix} \equiv |+\rangle_{\hat{n}} = \cos\left(\frac{\theta}{2}\right)e^{-i\varphi/2}|+\rangle + \sin\left(\frac{\theta}{2}\right)e^{i\varphi/2}|-\rangle$$

$-$ case: $$\begin{pmatrix} \cos\theta + 1 & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta + 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = 0 \rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \propto \begin{pmatrix} -\sin\theta e^{-i\varphi} \\ 1 + \cos\theta \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} c_1 \\ c_2 \end{pmatrix} \propto \begin{pmatrix} -2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)e^{-i\varphi} \\ 2\cos^2\left(\frac{\theta}{2}\right) \end{pmatrix}$$

$$\propto \begin{pmatrix} -\sin\left(\frac{\theta}{2}\right)e^{-i\varphi/2} \\ \cos\left(\frac{\theta}{2}\right)e^{i\varphi/2} \end{pmatrix} \equiv |-\rangle_{\hat{n}} = -\sin\left(\frac{\theta}{2}\right)e^{-i\varphi/2}|+\rangle + \cos\left(\frac{\theta}{2}\right)e^{i\varphi/2}|-\rangle$$