

Magnetic Resonance:

Suppose the magnetic field depends on time:

$$H = -\vec{M} \cdot \vec{B} = -\gamma \vec{S} \cdot \vec{B}(t).$$

In the Heisenberg representation

$$i\hbar \frac{d\vec{S}}{dt} = [\vec{S}, H(t)] = -\gamma [\vec{S}, \vec{S} \cdot \vec{B}].$$

Since $\vec{S} = \frac{\hbar}{2} \vec{\sigma}$ and $[\sigma_\alpha, \sigma_\beta] = 2i \epsilon_{\alpha\beta\gamma} \sigma_\gamma$,

$$\rightarrow [S_\alpha, S_\beta] = i\hbar \epsilon_{\alpha\beta\gamma} S_\gamma$$

$$\rightarrow i\hbar \frac{dS_\alpha}{dt} = -\gamma i\hbar \epsilon_{\alpha\beta\gamma} B_\beta S_\gamma$$

$$\rightarrow \frac{dS_\alpha}{dt} = -\gamma \epsilon_{\alpha\beta\gamma} B_\beta S_\gamma$$

$$\text{or } \frac{d\vec{S}}{dt} = -\gamma \vec{B} \times \vec{S} = \gamma \vec{S} \times \vec{B}$$

Let $\vec{B} = B_0 \hat{z}$ for starters.

$$\frac{d\langle S_x \rangle}{dt} = +\gamma B_0 \langle S_y \rangle$$

$$\frac{d\langle S_y \rangle}{dt} = -\gamma B_0 \langle S_x \rangle$$

$$\frac{d\langle S_z \rangle}{dt} = 0$$

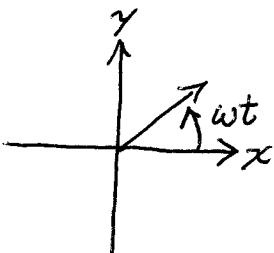
This has solution:

$$\langle S_x \rangle(t) = \cos(\omega_0 t) \langle S_x \rangle(0) - \sin(\omega_0 t) \langle S_y \rangle(0)$$

$$\langle S_y \rangle(t) = \cos(\omega_0 t) \langle S_y \rangle(0) + \sin(\omega_0 t) \langle S_x \rangle(0)$$

$$\langle S_z \rangle(t) = \langle S_z \rangle(0),$$

$$\text{where } \omega_0 = -\gamma B_0.$$



2.

Next consider $\vec{B} = B_0 \hat{z} + B_1 (\cos(\omega t) \hat{x} + \sin(\omega t) \hat{y})$.

$$\langle \vec{S} \rangle \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \langle S_x \rangle & \langle S_y \rangle & \langle S_z \rangle \\ B_1 \cos(\omega t) & B_1 \sin(\omega t) & B_0 \end{vmatrix}$$

$$= \hat{x} (\langle S_y \rangle B_0 - \langle S_z \rangle B_1 \sin(\omega t))$$

$$+ \hat{y} (\langle S_z \rangle B_1 \cos(\omega t) - \langle S_x \rangle B_0)$$

$$+ \hat{z} (\langle S_x \rangle B_1 \sin(\omega t) - \langle S_y \rangle B_1 \cos(\omega t))$$

$$\rightarrow \frac{d\langle S_x \rangle}{dt} = \underbrace{\gamma B_0}_{-\omega_0} \langle S_y \rangle - \underbrace{\gamma B_1 \sin(\omega t)}_{-\omega_1} \langle S_z \rangle$$

$$\frac{d\langle S_y \rangle}{dt} = -\gamma B_0 \langle S_x \rangle + \gamma B_1 \cos(\omega t) \langle S_z \rangle$$

$$\frac{d\langle S_z \rangle}{dt} = \gamma B_1 \sin(\omega t) \langle S_x \rangle - \gamma B_1 \cos(\omega t) \langle S_y \rangle$$

To solve these equations switch to a reference frame rotating at angular frequency ω :

$$S'_x \equiv \cos(\omega t) \langle S_x \rangle + \sin(\omega t) \langle S_y \rangle$$

$$S'_y \equiv \cos(\omega t) \langle S_y \rangle - \sin(\omega t) \langle S_x \rangle$$

$$S'_z \equiv \langle S_z \rangle.$$

$$\begin{aligned} \frac{dS'_x}{dt} &= -\omega \sin(\omega t) \langle S_x \rangle + \omega \cos(\omega t) \langle S_y \rangle \\ &\quad + \cos(\omega t) \{ \omega_0 \langle S_y \rangle + \omega_1 \sin(\omega t) \langle S_z \rangle \} \\ &\quad + \sin(\omega t) \{ +\omega_0 \langle S_x \rangle - \omega_1 \cos(\omega t) \langle S_z \rangle \} \\ &= -(\omega - \omega_0) \sin(\omega t) \langle S_x \rangle + (\omega - \omega_0) \cos(\omega t) \langle S_y \rangle \\ &= +(\omega - \omega_0) S'_y \end{aligned}$$

$$\begin{aligned} \frac{dS'_y}{dt} &= -\omega \sin(\omega t) \langle S_y \rangle - \omega \cos(\omega t) \langle S_x \rangle \\ &\quad + \cos(\omega t) \{ +\omega_0 \langle S_x \rangle - \omega_1 \cos(\omega t) \langle S_z \rangle \} \\ &\quad - \sin(\omega t) \{ -\omega_0 \langle S_y \rangle + \omega_1 \sin(\omega t) \langle S_z \rangle \} \\ &= -(\omega - \omega_0) \sin(\omega t) \langle S_y \rangle - (\omega - \omega_0) \cos(\omega t) \langle S_x \rangle - \omega_1 \langle S_z \rangle \\ &= -(\omega - \omega_0) S'_x - \omega_1 S'_z \end{aligned}$$

$$\frac{dS'_z}{dt} = +\omega_1 S'_y$$

Let $\Delta\omega = \omega - \omega_0$. Then

$$\frac{dS'_x}{dt} = \Delta\omega S'_y$$

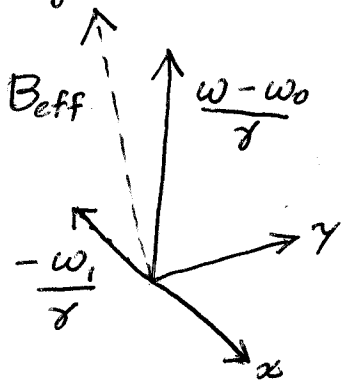
$$\frac{dS'_y}{dt} = -\Delta\omega S'_x - \omega_1 S'_z$$

$$\frac{dS'_z}{dt} = \omega_1 S'_y$$

$$\rightarrow \frac{d\vec{S}'}{dt} = \underbrace{-(\Delta\omega \hat{z} - \omega_1 \hat{x})}_{\gamma \vec{B}_{\text{eff}}} \times \vec{S}' = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ -\omega_1 & 0 & \Delta\omega \\ S'_x & S'_y & S'_z \end{vmatrix}$$

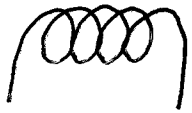
There is an effective field in the rotating frame of

$$\vec{B}_{\text{eff}} = \frac{1}{\gamma} \{ \Delta\omega \hat{z} - \omega_1 \hat{x} \}.$$



The spin precesses about \vec{B}_{eff} in the rotating frame.

NMR = nuclear magnetic resonance



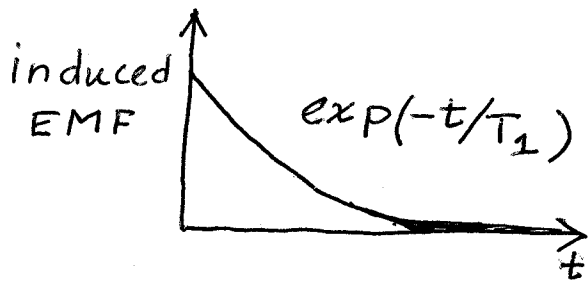
Radio frequency coil produces

$$B_1 \hat{x} \cos(\omega t) = \frac{1}{2} \{ B_1 \cos(\omega t) \hat{x} + B_1 \sin(\omega t) \hat{y} \} \\ + \frac{1}{2} \{ B_1 \cos(\omega t) \hat{x} - B_1 \sin(\omega t) \hat{y} \}.$$

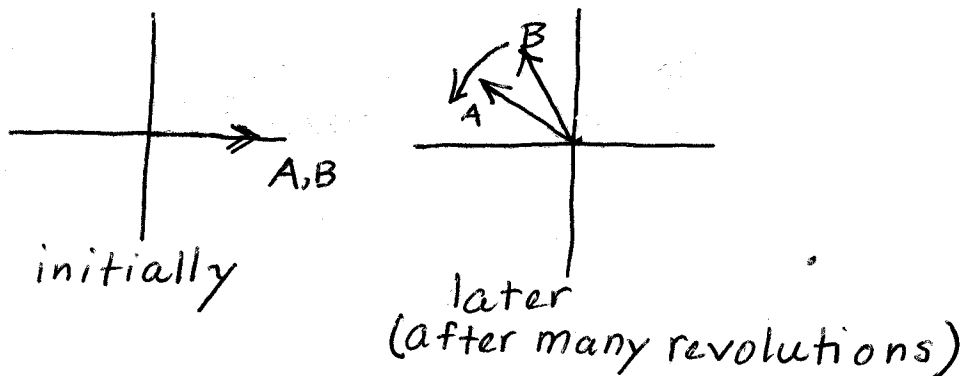
For $\omega \approx \omega_0$ apply a $\frac{\pi}{2}$ pulse, $\omega_1 t = \frac{\pi}{2} \rightarrow t = \frac{\pi}{2\omega_1}$.

↑ S'_z only goes to → S'_y only.

A precessing spin in the x - y plane produces an induced EMF in the coil, which can be detected. As the spins relax back to the \hat{z} direction the signal decays.



If B_0 is slightly non-uniform, then the signal will also decay because the spins in the x - y plane become dephased:



The time scale for this is called T_2 . It can be reversed by a π pulse.

