Commutators:

$$[x, p]\psi(x) = x \frac{\hbar}{i}\frac{d}{dx}\psi(x) - \frac{\hbar}{i}\frac{d}{dx} x \psi(x)$$

$$= x \frac{\hbar}{i}\frac{d\psi}{dx} - \frac{\hbar}{i}\psi(x) - x\frac{\hbar}{i}\frac{d\psi}{dx}$$

$$= i\hbar\, \psi(x)$$

$$\Rightarrow [x, p] = i\hbar$$

We can not measure x & p exactly at the same time.

Uncertainty principle:

$$\langle x\rangle = \langle \psi | x | \psi\rangle = \int dx\, \psi^*(x)\, x\, \psi(x) \quad , \text{where } \langle\psi|\psi\rangle = 1$$

$$\delta x \equiv x - \langle x\rangle$$

$$\langle p\rangle = \langle \psi | p | \psi\rangle = \int dx\, \psi^*(x) \frac{\hbar}{i}\frac{d\psi}{dx}$$

$$\delta p \equiv p - \langle p\rangle$$

δx and δp are operators. Let $\Delta x = \sqrt{\langle\psi|(\delta x)^2|\psi\rangle}$

$\Delta p = \sqrt{\langle\psi|(\delta p)^2|\psi\rangle}$.

Then $\boxed{\Delta x \Delta p \geq \frac{\hbar}{2}}$

Proof:

Consider $|\varphi\rangle = (\delta x + i\lambda\,\delta p)|\psi\rangle$, where $\lambda \in \mathbb{R}$.

$$\rightarrow \qquad \langle\varphi| = \langle\psi|(\delta x - i\lambda\delta p)$$

$$\rightarrow \langle\varphi|\varphi\rangle = \langle\psi|(\delta x - i\lambda\,\delta p)(\delta x + i\lambda\,\delta p)|\psi\rangle \geq 0$$

$$= \langle\psi|(\delta x)^2|\psi\rangle + \lambda^2\langle\psi|(\delta p)^2|\psi\rangle - i\lambda\langle\psi|[\delta p,\delta x]|\psi\rangle$$

But $[\delta p, \delta x] = [p - \langle p\rangle, x - \langle x\rangle] = [p, x] = -i\hbar$.

$$\rightarrow (\Delta x)^2 + \lambda^2(\Delta p)^2 - \lambda\hbar \geq 0 \quad \text{for all } \lambda \in \mathbb{R}.$$

$$\rightarrow (\Delta p)^2\left(\lambda^2 - \frac{\lambda\hbar}{(\Delta p)^2} + \frac{\hbar^2}{4(\Delta p)^4}\right) - \frac{\hbar^2}{4(\Delta p)^2} + (\Delta x)^2 \geq 0$$

$$\rightarrow \underbrace{(\Delta p)^2\left(\lambda - \frac{\hbar}{2(\Delta p)^2}\right)^2}_{\text{minimum value is 0 for } \lambda = \hbar/2(\Delta p)^2.} + (\Delta x)^2 - \frac{\hbar^2}{4(\Delta p)^2} \geq 0 \qquad (\star)$$

$$\rightarrow (\Delta x)^2 - \frac{\hbar^2}{4(\Delta p)^2} \geq 0 \rightarrow (\Delta x)^2 \geq \frac{\hbar^2}{4(\Delta p)^2} \rightarrow (\Delta x)^2(\Delta p)^2 \geq \frac{\hbar^2}{4}$$

$$\rightarrow \Delta x\,\Delta p \geq \frac{\hbar}{2} \quad //$$

This proof works for any two operators:

$$\Delta A\,\Delta B \geq \frac{1}{2}\left|\langle\psi|[A,B]|\psi\rangle\right|.$$