

Name:

**Exam 2 - PHY 4604 - Fall 2012**

November 16, 2012

8:20-10:20PM, NPB 1002

Directions: Please clear your desk of everything except for pencils and pens. The exam is closed book, and you are not allowed calculators or formula sheets. Leave substantial space between you and your neighbor. Show your work on the space provided on the exam. I can provide additional scratch paper if needed.

$$\begin{aligned}Y_0^0 &= \frac{1}{2}\sqrt{\frac{1}{\pi}} \\Y_1^1(\theta, \varphi) &= \frac{-1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{i\varphi} \cdot \sin \theta \\Y_1^0(\theta, \varphi) &= \frac{1}{2}\sqrt{\frac{3}{\pi}} \cdot \cos \theta \\Y_1^{-1}(\theta, \varphi) &= \frac{1}{2}\sqrt{\frac{3}{2\pi}} \cdot e^{-i\varphi} \cdot \sin \theta \\Y_2^2(\theta, \varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{2i\varphi} \cdot \sin^2 \theta \\Y_2^1(\theta, \varphi) &= \frac{-1}{2}\sqrt{\frac{15}{2\pi}} \cdot e^{i\varphi} \cdot \sin \theta \cdot \cos \theta \\Y_2^0(\theta, \varphi) &= \frac{1}{4}\sqrt{\frac{5}{\pi}} \cdot (3 \cos^2 \theta - 1) \\Y_2^{-1}(\theta, \varphi) &= \frac{1}{2}\sqrt{\frac{15}{2\pi}} \cdot e^{-i\varphi} \cdot \sin \theta \cdot \cos \theta \\Y_2^{-2}(\theta, \varphi) &= \frac{1}{4}\sqrt{\frac{15}{2\pi}} \cdot e^{-2i\varphi} \cdot \sin^2 \theta\end{aligned}$$

$$\begin{aligned}\sin(A \pm B) &= \sin(A) \cos(B) \pm \sin(B) \cos(A) \\ \cos(A \pm B) &= \cos(A) \cos(B) \mp \sin(A) \sin(B)\end{aligned}$$

1. Short answer section

- (a) What is the probability of measuring a non-degenerate eigenvalue,  $a_n$ , of an observable operator,  $A$ ? Let the state of the system be  $|\psi\rangle$  and the eigenvector of  $A$  with eigenvalue  $a_n$  be  $|\phi_n\rangle$ .

$$|\langle \phi_n | \psi \rangle|^2$$

- (b) What is the general uncertainty relation for two operators,  $A$  and  $B$ , which do not commute?

$$\sigma_A \sigma_B \geq \left| \frac{i}{2} \langle [A, B] \rangle \right|$$

- (c) What is the Hermitian conjugate of  $ABC$ , where  $A$ ,  $B$ , and  $C$  are three operators? (In other words what is  $(ABC)^\dagger$ ?)

$$(ABC)^\dagger = C^\dagger B^\dagger A^\dagger$$

- (d) What are the possible  $l$  values for the  $n = 3$  states of the Hydrogen atom?

$$l = 2, 1, 0$$

- (e) For each  $l$  in part (d), what are the possible  $m$  values? What is the degeneracy of the  $n = 3$  level of the Hydrogen atom, including the factor of two for spin?

$l = 2$	$m = -2, -1, 0, 1, 2$	degeneracy 5
$l = 1$	$m = -1, 0, 1$	3
$l = 0$	$m = 0$	1
		9 × 2 for spin = 18

## 2. Angular Momentum

(a) What is the commutator  $[L_+, L_-]$  expressed as a product of two operators?

$$\begin{aligned}[L_+, L_-] &= [L_x + iL_y, L_x - iL_y] \\ &= i[L_y, L_x] - i[L_x, L_y] = 2\hbar L_z\end{aligned}$$

$$\begin{aligned}[L_+, L_- L_-] &= [L_+, L_-] L_- + L_- [L_+, L_-] \\ &= 2\hbar (L_z L_- + L_- L_z)\end{aligned}$$

(b) For  $l = 2$  what is the matrix representation of the operator  $L_+ L_+ - L_- L_-$ ?

$$L_+ |2, -2\rangle = \hbar \sqrt{2(2+1) - (-2)(-2)} |2, -1\rangle = \hbar 2 |2, -1\rangle$$

$$L_+ |2, -1\rangle = \hbar \sqrt{2(2+1) - (-1)(-1)} |2, 0\rangle = \hbar \sqrt{6} |2, 0\rangle$$

$$L_+ |2, 0\rangle = \hbar \sqrt{2(2+1) - 0(1+0)} |2, 1\rangle = \hbar \sqrt{6} |2, 1\rangle$$

$$L_+ |2, 1\rangle = \hbar \sqrt{2(2+1) - 1(1+1)} |2, 2\rangle = \hbar 2 |2, 2\rangle$$

$$(L_+)^2 |2, -2\rangle = \hbar^2 2\sqrt{6} |2, 0\rangle$$

$$(L_+)^2 |2, -1\rangle = \hbar^2 6 |2, 1\rangle$$

$$(L_+)^2 |2, 0\rangle = \hbar^2 2\sqrt{6} |2, 2\rangle$$

$$\rightarrow (L_+)^2 = \frac{1}{\hbar^2} \begin{pmatrix} 0 & 0 & 2\sqrt{6} & 0 & 0 \\ 0 & 0 & 6 & 0 & 0 \\ 0 & 0 & 0 & 2\sqrt{6} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \quad (L_-)^2 = (L_+^\dagger)^2$$

$$\rightarrow (L_+)^2 - (L_-)^2$$

$$\begin{pmatrix} 0 & 0 & 2\sqrt{6} & 0 & 0 \\ 0 & 0 & 0 & 6 & 0 \\ -2\sqrt{6} & 0 & 0 & 0 & 2\sqrt{6} \\ 0 & -6 & 0 & 0 & 0 \\ 0 & 0 & -2\sqrt{6} & 0 & 0 \end{pmatrix}$$

- (c) Express the  $l = 2$  spherical harmonics in terms of  $x/r$ ,  $y/r$ , and  $z/r$ . Which linear combination of the spherical harmonics gives an angular dependence of  $(x^2 - y^2)/r^2$ ?

$$\frac{x}{r} = \sin\theta \cos\varphi ; \frac{y}{r} = \sin\theta \sin\varphi ; \frac{z}{r} = \cos\theta$$

$$e^{\pm i\varphi} = \cos\varphi \pm i\sin\varphi$$

$$e^{\pm 2i\varphi} = \cos 2\varphi \pm i\sin 2\varphi$$

$$= (\cos^2\varphi - \sin^2\varphi) \pm i 2\sin\varphi \cos\varphi$$

$$Y_{2,2} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \left\{ \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 + i 2 \left(\frac{x}{r}\right)\left(\frac{y}{r}\right) \right\}$$

$$Y_{2,1} = \frac{-1}{2} \sqrt{\frac{15}{2\pi}} \left\{ \frac{x}{r} + i \frac{y}{r} \right\} \left(\frac{z}{r}\right)$$

$$Y_{2,0} = \frac{1}{4} \sqrt{\frac{5}{\pi}} \left( 3 \left(\frac{z}{r}\right)^2 - 1 \right)$$

$$Y_{2,-1} = \frac{1}{2} \sqrt{\frac{15}{2\pi}} \left\{ \frac{x}{r} - i \frac{y}{r} \right\} \left(\frac{z}{r}\right)$$

$$Y_{2,-2} = \frac{1}{4} \sqrt{\frac{15}{2\pi}} \left\{ \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 - 2i \left(\frac{x}{r}\right)\left(\frac{y}{r}\right) \right\}$$

$$Y_{2,2} + Y_{2,-2} = \frac{1}{2} \sqrt{\frac{15}{2\pi}} \left\{ \left(\frac{x}{r}\right)^2 - \left(\frac{y}{r}\right)^2 \right\}$$

### 3. Measurements

Two linear operators are represented by the matrices

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 2 \end{pmatrix} \text{ and } B = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 0 \end{pmatrix}.$$

(a) What are the eigenvalues and eigenvectors of  $A$  and  $B$ ?

	eigenvalue	eigenvector
$A:$	2	$\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$
	1	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$
	-1	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

$B:$	2	$\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$
	1	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$
	-1	$\frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$

- (b) A measurement of  $B$  is performed and its smallest eigenvalue is observed. Next a measurement of  $A$  is performed. What are the possible outcomes and their associated probabilities?

$$\text{For } \lambda_B = -1, |\psi\rangle = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}.$$

Possible  
outcomes

Probability

2

$$\left| (0 \ 0 \ 1) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right|^2 = \frac{1}{2}$$

1

$$\left| \frac{1}{\sqrt{2}} (1 \ 1 \ 0) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right|^2 = \frac{1}{4}$$

-1

$$\left| \frac{1}{\sqrt{2}} (1 \ -1 \ 0) \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right|^2 = \frac{1}{4}$$

- (c) The measurement of  $A$  yields the largest eigenvalue. What are the possible outcomes and the associated probabilities of a subsequent measurement of  $B$ ?

$$\text{For } \lambda_A = 2, |\psi\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

Possible  
outcomes

Probability

2

$$\left| (0 \ 1 \ 0) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = 0$$

1

$$\left| \frac{1}{\sqrt{2}} (1 \ 0 \ 1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \frac{1}{2}$$

-1

$$\left| \frac{1}{\sqrt{2}} (1 \ 0 \ -1) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right|^2 = \frac{1}{2}$$

#### 4. Radial Schrodinger Equation

In the following consider the spherically symmetric potential  $V(r) = V_0 > 0$  for  $0 < r < r_0$  and  $V(r) = 0$  for  $r > r_0$ .

- (a) Solve the radial Schrodinger equation for  $u(r)$  in the region  $0 < r < r_0$  with the above  $V(r)$  and  $l = 0$ . Assume that  $0 < E < V_0$ . Apply the boundary condition at  $r = 0$  to reduce this to one solution.

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + V_0 u = E u \rightarrow \frac{d^2 u}{dr^2} = \frac{2m}{\hbar^2} (V_0 - E) u$$

$$\text{Let } \rho = \sqrt{\frac{2m}{\hbar^2} (V_0 - E)}. \quad u(r) = A e^{\rho r} + B e^{-\rho r}.$$

$$u(0) = A + B = 0 \rightarrow u(r) = A(e^{\rho r} - e^{-\rho r}).$$

$$\text{or } u(r) = A' \sinh(\rho r); A' = 2A$$

- (b) Solve the radial Schrodinger equation for  $u(r)$  in the region  $r > r_0$  again assuming that  $0 < E < V_0$  and  $l = 0$ .

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} = E u \rightarrow \frac{d^2 u}{dr^2} = -\frac{2mE}{\hbar^2} u$$

$$\text{Let } k = \sqrt{\frac{2mE}{\hbar^2}}. \quad u(r) = B e^{i k r} + C e^{-i k r}$$

- (c) What are the equations for the boundary conditions at  $r = r_0$  for the wave function  $u(r)$  given in parts (a) and (b)?

$$u(r_0) = A' \sinh(\rho r_0) = B e^{i k r_0} + C e^{-i k r_0}$$

$$u'(r_0) = \rho A' \cosh(\rho r_0) = i k (B e^{i k r_0} - C e^{-i k r_0})$$

- (d) Solve for  $u(r)$  up to an overall multiplicative constant.

$$A' \begin{pmatrix} \sinh(\rho r_0) \\ \frac{\rho}{i k} \cosh(\rho r_0) \end{pmatrix} = \begin{pmatrix} e^{i k r_0} & e^{-i k r_0} \\ e^{i k r_0} & -e^{-i k r_0} \end{pmatrix} \begin{pmatrix} B \\ C \end{pmatrix}$$

$$\begin{pmatrix} B \\ C \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e^{-i k r_0} & e^{-i k r_0} \\ e^{i k r_0} & -e^{i k r_0} \end{pmatrix} \begin{pmatrix} \sinh(\rho r_0) \\ \frac{\rho}{i k} \cosh(\rho r_0) \end{pmatrix} A'$$

$$u(r < r_0) = A' \sinh(\rho r)$$

$$u(r > r_0) = \frac{A'}{2} \left( \sinh(\rho r_0) + \frac{\rho}{i k} \cosh(\rho r_0) \right) e^{i k (r - r_0)} \\ + \frac{A'}{2} \left( \sinh(\rho r_0) - \frac{\rho}{i k} \cosh(\rho r_0) \right) e^{-i k (r - r_0)}$$

$$= A' \sinh(\rho r_0) \cos(k(r - r_0))$$

$$+ A' \frac{\rho}{k} \cosh(\rho r_0) \sin(k(r - r_0))$$



### 5. Dimensional analysis

For the three dimensional Schrodinger equation with the spherically symmetric potential  $V(r) = \alpha r$ , use dimensional analysis to estimate the characteristic length and energy scale of the lowest energy  $l = 0$  solution. (It will be easiest to use the  $u(r)$  equation.)

$$-\frac{\hbar^2}{2m} \frac{d^2 u}{dr^2} + \alpha r u = E u$$

$$-\frac{d^2 u}{dr^2} + \frac{2m\alpha r}{\hbar^2} u = \frac{2mE}{\hbar^2} u$$

$$\frac{1}{L^2}$$

$$\frac{1}{L^2} = \frac{2m\alpha}{\hbar^2} L$$

$$\frac{2mE}{\hbar^2} = \frac{1}{L^2}$$

$$L = \left( \frac{\hbar^2}{2m\alpha} \right)^{1/3}$$

$$E = \frac{\hbar^2}{2mL^2} = \frac{\hbar^2}{2m} \left( \frac{2m\alpha}{\hbar^2} \right)^{2/3} = \left( \frac{\hbar^2}{2m} \right)^{1/3} \alpha^{2/3}$$